RAMSEY THEORY FOR p-QUASICYCLIC GROUPS WITH A VIEW TOWARDS TOPOLOGICAL DYNAMICS

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Abstract. We prove additive and multiplicative partition theorems, obtaining combinatorial results for the p-quasicyclic group $\mathbb{Z}(p^\infty)$, where p is a prime number. We also get density results for $\mathbb{Z}(p^\infty)$ via left Følner sequences of non-empty finite subsets of it, giving a sufficient condition in order a subset of $\mathbb{Z}(p^\infty)$ to contain arbitrary long arithmetic progressions. Finally, we introduce the notion of a $\mathbb{Z}(p^\infty)$-dynamical system extending the classical notion of a topological dynamical system and we prove (multiple) recurrent results for the p-quasicyclic groups. In particular, we prove recurrent results analogous to Furstenberg-Weiss type theorems for classical systems.

Introduction

Let p be a prime number. The *p-quasicyclic* group (or *Prüfer p-group*) is defined to be the Sylow p-subgroup of $\mathbb{Q}/\mathbb{Z}$ (i.e. the unique maximal p-subgroup of $\mathbb{Q}/\mathbb{Z}$), that is, the set of all elements of $\mathbb{Q}/\mathbb{Z}$ whose order is a power of p. This group is denoted by $(\mathbb{Z}(p^\infty), +)$. So, $\mathbb{Z}(p^\infty)$ is the set of all real numbers q which can be written uniquely in the form

$$q = \sum_{t=1}^{\infty} d_t p^{-t}$$

where $(d_t)_{t\in\mathbb{N}} \subseteq \mathbb{N} \cup \{0\}$ with $0 \leq d_t \leq p-1$ for every $t > 0$ and $d_t = 0$ for all but finite many t. We call the set $\{t_1 < \ldots < t_l\} = \{t \in \mathbb{N} : \ d_t \neq 0\}$ the *domain* of q. Any group isomorphic to it (for example every infinite locally cyclic p-group), also called p-quasicyclic group (for the terminology, notation and a characterization, see [ZML]). The subgroup structure of $\mathbb{Z}(p^\infty)$ is particulary simple: all proper subgroups are finite and cyclic, and there is exactly one of order p^n for each $n \in \mathbb{N}$. In particular, this means that the subgroups are fully invariant and linearly ordered by inclusion. p-quasicyclic groups play an important role in the infinite abelian group theory since they share a lot of properties. For example p-quasicyclic groups are locally cyclic, divisale, co-Hopfian, they are the only infinite groups with a linearly ordered subgroup lattice and they are

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also the only infinite solvable groups whose proper subgroups are all finite. They can also be defined (up to isomorphism) in a number of equivalent ways:

1) A p-quasicyclic group is the group of all p^n-th complex roots of 1, for all $n \in \mathbb{N}$.

2) A p-quasicyclic group is the injective hull of $\mathbb{Z}/p\mathbb{Z}$ (viewing abelian groups as $\mathbb{Z}$-modules).

3) A p-quasicyclic group is the direct limit of the groups $\mathbb{Z}/p^n\mathbb{Z}$.

In our work, we will identify every element $q \in \mathbb{Z}(p^\infty)$ with a located word $d_{t_1} \dots d_{t_l}$ over the alphabet $\Sigma = \{1, \dots, p-1\}$ (a located word over Σ is a function from a finite subset of natural numbers to the alphabet Σ) where $\{t_1 < \dots < t_l\}$ is its domain (see §1). Using a partition theorem (Theorem 1.1) proved in [BBH] for located words, we obtain:

(1) an additive (resp. a multiplication) partition theorem for the group $\mathbb{Z}(p^\infty)$ in Theorem 1.2 (resp. in Theorem 1.3).

Theorem 1.2 is our starting point for this paper and we will use it to obtain topological dynamics results concerning $\mathbb{Z}(p^\infty)$ in sections $3-6$. More specifically:

(2) we extend Theorems 1.2 and 1.3 providing a simultaneously additive and multiplicative partition result for $\mathbb{Z}(p^\infty)$ in Theorem 1.9, mostly referring to [BHL] since the arguments are completely similar to the ones concerning the dyadic rationals; and

(3) we provide a counterexample (Theorem 1.11) which shows that there exists a finite partition of $\mathbb{Z}(p^\infty)$ such that we cannot find a sequence $(x_n)_{n\in\mathbb{N}}$ such that all its finite sums and products are monochromatic (as in [BHL] for the dyadic rational numbers in $(0, 1)$).

Via the existence of left Følner sequences in the set of non-empty finite subsets of $\mathbb{Z}(p^\infty)$ (Definition 2.6), since $(\mathbb{Z}(p^\infty), +)$ is a countable abelian (semi)group (see §2), we prove:

(4) Szemerédi type theorems (Theorem 2.8) for the group $\mathbb{Z}(p^\infty)$ giving a sufficient condition via the Følner sequences in order a subset of $\mathbb{Z}(p^\infty)$ to contain arbitrary long arithmetic progressions; and

(5) density Halles-Jewett type theorems (Theorem 2.12) for $\mathbb{Z}(p^\infty)$, using fundamental results from [FuKa], [FuKa2], [HS] and the representation of the elements of $\mathbb{Z}(p^\infty)$ as located words.

Extending the classical notion of the topological dynamical system, we introduce the notion of a $\mathbb{Z}(p^\infty)$-dynamical system (Definition 4.1). Consequently, we develop a recurrence theory for $\mathbb{Z}(p^\infty)$-dynamical systems (for recurrent results concerning the set of rational numbers see [K]).

The classical results of the theory of topological dynamical systems proved by Furstenberg and Weiss are the following:

Theorem 0.1. *(*[FuW]*, 1978) If X is a compact metric space and $T_1, \dots, T_l$ are commuting continuous maps of X to itself, then there exists a point $x \in X$ and a sequence $(\alpha_n)_{n\in\mathbb{N}} \subseteq \mathbb{N}$, $\alpha_n \to \infty$ such that $T_i^{\alpha_n}(x) \to x$ simultaneously for $1 \leq i \leq l$ (in this case, x is said to be a multiple recurrent point for $T_1, \dots, T_l$).*

A consequence (in fact an equivalent form) of Theorem 0.1 is the following:

Theorem 0.2. *Let $l \in \mathbb{N}$ and $\varepsilon > 0$. If X is a compact metric space and $T_1, \dots, T_l$ are commuting continuous maps of X to itself, then there exists $x_0 \in X$ and $n_0 \in \mathbb{N}$, such that $T_i^{n_0}(x_0) \in B(x_0, \varepsilon)$ for every $1 \leq i \leq l$.*

Theorem 0.2, which is the multiple version of Birkhoff's recurrence theorem ([Bi]), can be considered as the topological dynamics version of Gallai's partition theorem (see [GRS]), which is the multidimensional extension of van der Waerden's theorem ([vdW]), that for any finite partition of the set of natural numbers there exists a cell of the partition which contains arbitrary long arithmetic progressions.

We develop a recurrence theory for $\mathbb{Z}(p^\infty)$-dynamical systems, extending the fundamental results of Furstenberg and Weiss which concern dynamical systems indexed by natural numbers (see [Fu], [FuW]). More specifically we obtain:

(6) a topological partition theorem for p-quasicyclic groups (Theorem 3.2) and its multiple version (Theorem 5.1) extending Theorem 0.2 to $\mathbb{Z}(p^\infty)$-dynamical systems.

(7) Introducing the minimal $\mathbb{Z}(p^\infty)$-systems and characterizing them as systems having only uniformly $\mathbb{Z}(p^\infty)$-recurrent points we prove, in Theorem 5.2, a strong recurrence property of minimal $\mathbb{Z}(p^\infty)$-dynamical systems giving an equivalent reformulation of Theorem 5.1.

(8) We obtain a strengthening of Theorem 0.1 for $\mathbb{Z}(p^\infty)$-dynamical systems in Theorem 6.1 and we prove that it is an equivalent expression of Theorems 5.1 and 5.2.

The essentially stronger nature of the partition and topological dynamics theory developed in this paper makes it reasonable to expect that it will find substantial applications

in (Ramsey) ergodic theory, Analysis, Topology and in various other branches of mathematics. Also, since p-quasicyclic groups play an important role in the infinite abelian group theory, as mentioned before, it is believed that this paper will give new and interesting results in group theory and more generally in Algebra.

We will use the following notation.

Notation. Let $\mathbb{N}=\{1,2,\dots\}$ be the set of natural numbers, $\mathbb{Z}=\{\dots,-2,-1,0,1,2,\dots\}$ the set of integer numbers, $\mathbb{Q}=\{\frac{m}{n}:\ m\in\mathbb{Z},\ n\in\mathbb{N}\}$ the set of rational numbers and $\mathbb{Z}(p^\infty)^*=\mathbb{Z}(p^\infty)\setminus\{0\}$ the non-zero elements of $\mathbb{Z}(p^\infty)$. By $|A|$ we denote the cardinality of a finite set A. Let $[X]^{<\omega}_{>0}$ be the set of all the non-empty finite subsets of X. For a sequence $(x_n)_{n\in\mathbb{N}}$ of real numbers we set

$FS\big[(x_n)_{n\in\mathbb{N}}\big]=\{\sum_{n\in F}x_n:\ F\in[\mathbb{N}]^{<\omega}_{>0}\}$,
$FP\big[(x_n)_{n\in\mathbb{N}}\big]=\{\prod_{n\in F}x_n:\ F\in[\mathbb{N}]^{<\omega}_{>0}\}$, and
$PP\big[(x_n)_{n\in\mathbb{N}}\big]=\{x_nx_m:\ n\neq m\in\mathbb{N}\}$.

1. Coloring results for p-quasicyclic groups

In this paragraph we are dealing with the combinatorial structure of the set $\mathbb{Z}(p^\infty)$, proving coloring results for it. In order to do so, firstly, we will recall the notion of the (constant and variable) located words over a finite alphabet (from [BBH]).

For a finite alphabet $\Sigma=\{\alpha_1,\alpha_2,\dots,\alpha_k\}$ let the sets of *located words* over Σ to be

$L(\Sigma)=\{w=w_{n_1}\dots w_{n_l}:\ l\in\mathbb{N}, n_1<\dots<n_l\in\mathbb{N}, w_{n_i}\in\Sigma$ for every $1\leq i\leq l\}$,
$L(\Sigma;\upsilon)=\{w=w_{n_1}\dots w_{n_l}:\ l\in\mathbb{N}, n_1<\dots<n_l\in\mathbb{N}, w_{n_i}\in\Sigma\cup\{\upsilon\}$ for every $1\leq i\leq l$
and there exists $1\leq i\leq l$ with $w_{n_i}=\upsilon\}$, where $\upsilon\notin\Sigma$ be a *variable*, and
$L(\Sigma\cup\{\upsilon\})=L(\Sigma)\cup L(\Sigma;\upsilon)$.

For $w=w_{n_1}\dots w_{n_l}, u=u_{s_1}\dots u_{s_m}\in L(\Sigma\cup\{\upsilon\})$ we write $w<_R u\iff n_l<s_1$ and we set $w\star u=w_{n_1}\dots w_{n_l}u_{s_1}\dots u_{s_m}$.

Let $L^\infty(\Sigma;\upsilon)=\{(w_n)_{n\in\mathbb{N}}\subseteq L(\Sigma;\upsilon): w_n<_R w_{n+1}$ for every $n\in\mathbb{N}\}$.

For every $m\in\mathbb{N}\cup\{0\}$ we define the functions $T_m: L(\Sigma\cup\{\upsilon\})\to L(\Sigma\cup\{\upsilon\})$ setting for $w=w_{n_1}\dots w_{n_l}\in L(\Sigma\cup\{\upsilon\})$: $T_0(w)=w$ and, for $m\in\mathbb{N}$, $T_m(w)=u_{n_1}\dots u_{n_l}$, where, for $1\leq i\leq l$, $u_{n_i}=w_{n_i}$ if $w_{n_i}\in\Sigma$, $u_{n_i}=\alpha_m$ if $w_{n_i}=\upsilon$ and $m\leq k$ and finally $u_{n_i}=\alpha_k$ if $w_{n_i}=\upsilon$ and $m>k$.

Let $\vec{w}=(w_n)_{n\in\mathbb{N}}\in L^\infty(\Sigma;\upsilon)$. The set $EV(\vec{w})$ of all the *extracted variable located words* of $\vec{w}$ consists of the words of the form $T_{m_1}(w_{n_1})\star\dots\star T_{m_\lambda}(w_{n_\lambda})$, where $\lambda\in\mathbb{N}$, $n_1<\dots<n_\lambda\in\mathbb{N}$ and $m_1,\dots,m_\lambda\in\mathbb{N}\cup\{0\}$ such that $0\leq m_i\leq k$ for every $1\leq i\leq\lambda$ and $0\in\{m_1,\dots,m_\lambda\}$, and the set $E(\vec{w})$ of all the *extracted located words* of $\vec{w}$ consists

of the words of the form $T_{m_1}(w_{n_1})\star\ldots\star T_{m_\lambda}(w_{n_\lambda})$, where $\lambda\in\mathbb{N}$, $n_1<\ldots<n_\lambda\in\mathbb{N}$ and $m_1,\ldots,m_\lambda\in\mathbb{N}$ such that $1\leq m_i\leq k$ for every $1\leq i\leq\lambda$.

Theorem 1.1 (Partition theorem for located words([BBH])). *Let $\Sigma=\{\alpha_1,\alpha_2\ldots,\alpha_k\}$ be a finite alphabet, $\upsilon\notin\Sigma$ a variable and $r,s\in\mathbb{N}$. If $L(\Sigma;\upsilon)=A_1\cup\ldots\cup A_r$ and $L(\Sigma)=C_1\cup\ldots\cup C_s$, then there exists a sequence $\vec{w}=(w_n)_{n\in\mathbb{N}}\in L^\infty(\Sigma;\upsilon)$ and $1\leq i_0\leq r$, $1\leq j_0\leq s$ such that*

$$EV(\vec{w})\in A_{i_0}\ \textit{and}\ E(\vec{w})\in C_{j_0}.$$

In order to derive a coloring result for p-quasicyclic goups, we will use Theorem 1.1 identifying every element of $\mathbb{Z}(p^\infty)$ with a located word.

Let p be a prime number. Every element q of $\mathbb{Z}(p^\infty)$ has a unique expression in the form

$$q=\sum_{t=1}^{\infty}d_tp^{-t}$$

where $(d_t)_{t\in\mathbb{N}}\subseteq\mathbb{N}\cup\{0\}$ with $0\leq d_t\leq p-1$ for every $t>0$ and $d_t=0$ for all but finite many t. So, for every non-zero element $q\in\mathbb{Z}(p^\infty)$, there exists a unique $l\in\mathbb{N}$, $\{t_1<\ldots<t_l\}=dom(q)\in[\mathbb{N}]^{<\omega}_{>0}$ and $\{d_{t_1},\ldots,d_{t_l}\}\subseteq\{1,\ldots,p-1\}$ such that

$$q=\sum_{t\in dom(q)}d_tp^{-t}.$$

Observe that

$$0<\sum_{t\in dom(q)}d_tp^{-t}<1\ \text{ if }\ dom(q)\neq\emptyset.$$

For two non-zero elements $q_1,q_2\in\mathbb{Z}(p^\infty)$ we set

$$q_1\prec q_2\iff\max dom(q_1)<\min dom(q_2).$$

Using the previous representation and the Theorem 1.1 we have the following result concerning finite partitions of the group $\mathbb{Z}(p^\infty)$:

Theorem 1.2. *Let $\mathbb{Z}(p^\infty)=Z_1\cup\ldots\cup Z_r$ for $r\in\mathbb{N}$. Then, there exist $1\leq i_0\leq r$ and for every $n\in\mathbb{N}$ a function $p_n:\{0,1,\ldots,p-1\}\to\mathbb{Z}(p^\infty)$ with*

$$p_n(i)=\sum_{t\in C_n}d^n_tp^{-t}+i\sum_{t\in V_n}p^{-t},$$

where $C_n,V_n\in[\mathbb{N}]^{<\omega}_{>0}$ with $C_n\cap V_n=\emptyset$, $d^n_t\in\mathbb{N}$ with $1\leq d^n_t\leq p-1$ for $t\in C_n$, which satisfy $p_n(i_n)\prec p_{n+1}(i_{n+1})$ for every $n\in\mathbb{N}$, and

$$FS\big[\big(p_n(i_n)\big)_{n\in\mathbb{N}}\big]\subseteq Z_{i_0}$$

for all $0\leq i_n\leq p-1$, for every $n\in\mathbb{N}$.

Proof. Let $\Sigma = \{1, 2, \ldots, p-1\}$. For $\upsilon = 0$ we define the function $g : L(\Sigma \cup \{0\}) \to \mathbb{Z}(p^\infty)$, which sends a word $w = d_{t_1} \ldots d_{t_l} \in L(\Sigma \cup \{0\})$ to the element

$$g(w) = \textstyle\sum_{t \in dom(w)} d_t p^{-t}.$$

It is easy to see that the restriction of g to the set of the constant words $L(\Sigma)$ is one-to-one and onto $\mathbb{Z}(p^\infty)^*$ and that $g(w_1 \star w_2) = g(w_1) + g(w_2)$ for every $w_1 <_R w_2 \in L(\Sigma \cup \{0\})$. Also, observe that, via the function g, each variable word $w = d_{t_1} \ldots d_{t_l} \in L(\Sigma; 0)$ corresponds to a function p which sends every $1 \leq i \leq p-1$ to

$$p(i) = g(T_i(w)) = \sum_{t \in C} d_t p^{-t} + i \sum_{t \in V} p^{-t},$$

where $C = \{t \in dom(w) : \ d_t \in \Sigma\}$ and $V = \{t \in dom(w) : \ d_t = 0\}$.

According to Theorem 1.1 there exist a sequence $\vec{w} = (w_n)_{n \in \mathbb{N}} \in L^\infty(\Sigma; 0)$ and an index $1 \leq i_0 \leq r$ such that $E(\vec{w}) \subseteq g^{-1}(Z_{i_0})$. Set $p_n : \{1, \ldots, p-1\} \to \mathbb{Z}(p^\infty)$ with $p_n(i) = g(T_i(w_{2n}) \star T_1(w_{2n+1}))$, for every $n \in \mathbb{N}$. The functions p_n have the required properties. □

Seeing the set $\mathbb{Z}(p^\infty)$ with the multiplication in stead of addition we have that $(\mathbb{Z}(p^\infty), \cdot)$ is a semigroup. Using Theorem 1.2 via the function $h : (\mathbb{Z}(p^\infty), +) \to (\mathbb{Z}(p^\infty), \cdot)$ with

$$h(\textstyle\sum_{t=1}^{\infty} d_t p^{-t}) = \prod_{t=1}^{\infty} d_t p^{-t},$$

we have the following result:

Theorem 1.3. *Let $\mathbb{Z}(p^\infty) = Z_1 \cup \ldots \cup Z_r$ for $r \in \mathbb{N}$. Then, there exist $1 \leq i_0 \leq r$ and for every $n \in \mathbb{N}$ a function $s_n : \{0, 1, \ldots, p-1\} \to \mathbb{Z}(p^\infty)$ with*

$$s_n(i) = \prod_{t \in C_n} d_t^n p^{-t} \cdot i \prod_{t \in V_n} p^{-t},$$

where $C_n, V_n \in [\mathbb{N}]^{<\omega}_{>0}$ with $C_n \cap V_n = \emptyset$, $d_t^n \in \mathbb{N}$ with $1 \leq d_t^n \leq p-1$ for $t \in C_n$, which satisfy $\max(C_n \cup V_n) < \min(C_{n+1} \cup V_{n+1})$ *for every $n \in \mathbb{N}$, and*

$$FP\big[\big(s_n(i_n)\big)_{n \in \mathbb{N}}\big] \subseteq Z_{i_0}$$

for all $0 \leq i_n \leq p-1$, for every $n \in \mathbb{N}$.

Proof. According to Theorem 1.2 there exist $1 \leq i_0 \leq r$ and for every $n \in \mathbb{N}$ a function $p_n : \{0, 1, \ldots, p-1\} \to \mathbb{Z}(p^\infty)$ with

$$p_n(i) = \sum_{t \in C_n} d_t^n p^{-t} + i \sum_{t \in V_n} p^{-t},$$

where $C_n, V_n \in [\mathbb{N}]^{<\omega}_{>0}$ with $C_n \cap V_n = \emptyset$, $d_t^n \in \mathbb{N}$ with $1 \leq d_t^n \leq p-1$ for $t \in C_n$, which satisfy $p_n(i_n) \prec p_{n+1}(i_{n+1})$ for every $n \in \mathbb{N}$, and

$$FS\big[\big(p_n(i_n)\big)_{n\in\mathbb{N}}\big] \subseteq h^{-1}(Z_{i_0})$$

for all $0 \leq i_n \leq p-1$, for every $n \in \mathbb{N}$. Setting $s_n = h(p_n)$ for every $n \in \mathbb{N}$ we have that

$$FP\big[\big(s_n(i_n)\big)_{n\in\mathbb{N}}\big] = FP\big[\big(h(p_n(i_n))\big)_{n\in\mathbb{N}}\big] = h(FS\big[\big(p_n(i_n)\big)_{n\in\mathbb{N}}\big]) \subseteq Z_{i_0}. \qquad \square$$

Theorems 1.2 and 1.3 informs us that for any partition of $\mathbb{Z}(p^\infty)$ into finite many sets, we can find sequences $(x_n)_{n\in\mathbb{N}}$, $(y_n)_{n\in\mathbb{N}}$ such that the sets $FS((x_n)_{n\in\mathbb{N}})$ and $FP((y_n)_{n\in\mathbb{N}})$ are monochromatic respectively. One may wander if we can have these results simultaneously for any partition of $\mathbb{Z}(p^\infty)$.

We will now see some more partition properties of the group $\mathbb{Z}(p^\infty)$, identifying the set $\mathbb{Z}(p^\infty)^*$ with the set $\mathbb{Q}_p \cap (0,1)$, where $\mathbb{Q}_p = \{\frac{a}{p^n} : \ n \in \mathbb{N}, \ \alpha \in \mathbb{N}\}$ is the set of the p-adic rationals. We will sketch the method of the proof of a simultaneously additive and multiplicative partition result in $\mathbb{Z}(p^\infty)$, introduced in [BHL]. We will present only some arguments, referring to [BHL], since the arguments for $\mathbb{Q}_p \cap (0,1)$ are analogous to the ones of Bergelson, Hindman and Leader (in [BHL], §$3-5$) for the set $\mathbb{Q}_2$ of the dyadic rational numbers.

We will need some terminology from the theory of ultrafilters.

Ultrafilters. Let X be a non-empty set. An *ultrafilter* on the set X is a zero-one finite additive measure μ defined on all the subsets of X. The set of all ultrafilters on the set X is denoted by βX. So, $\mu \in \beta X$ if and only if

(i) $\mu(A) \in \{0,1\}$ for every $A \subseteq X$, $\mu(X) = 1$, and
(ii) $\mu(A \cup B) = \mu(A) + \mu(B)$ for every $A, B \subseteq X$ with $A \cap B = \emptyset$.

For $\mu \in \beta X$, it is easy to see that $\mu(A \cap B) = 1$, if $\mu(A) = 1$ and $\mu(B) = 1$. For every $x \in X$ is defined the *principal ultrafilter* μ_x on X which corresponds a set $A \subseteq X$ to $\mu_x(A) = 1$ if $x \in A$ and $\mu_x(A) = 0$ if $x \notin A$. So, μ is a non-principal ultrafilter on X if and only if $\mu(A) = 0$ for every finite subset A of X.

The set βX becomes a compact Hausdorff space if it be endowed with the topology $\mathfrak{T}$ which has basis the family $\{\overline{A} : A \subseteq X\}$, where $\overline{A} = \{\mu \in \beta X : \mu(A) = 1\}$. It is easy to see that $\overline{A \cap B} = \overline{A} \cap \overline{B}$, $\overline{A \cup B} = \overline{A} \cup \overline{B}$ and $\overline{X \setminus A} = \beta X \setminus \overline{A}$ for every $A, B \subseteq X$. We always consider the set βX endowed with the topology $\mathfrak{T}$. Also βX is called the *Stone-Čech compactification* of the set X.

If $(X,+)$ is a semigroup, then a binary operation $+$ is defined on βX, extending the operation $+$ on X, corresponding to every $\mu_1, \mu_2 \in \beta X$ the ultrafilter $\mu_1+\mu_2 \in \beta X$, with

$$(\mu_1+\mu_2)(A) = \mu_1(\{x \in X : \mu_2(\{y \in X : x+y \in A\}) = 1\}) \text{ for every } A \subseteq X.$$

With this operation $(\beta X, +, \mathfrak{T})$ becomes a right topological semigroup, that is, for every $\mu \in \beta X$ the function $f_{x_0} : \beta X \to \beta X$ with $f_{x_0}(\mu) = \mu_{x_0} + \mu$ is continuous.

As in [BHL], we define the columns condition introduced by Rado ([Ra]) in his characterization of partition regularity of homogeneous equations.

Definition 1.4. Let $u, v \in \mathbb{N}$, C a $u \times v$ matrix entries from $\mathbb{R}$ and let R be a subring of $\mathbb{R}$. Then C satisfies the *columns condition* over R if the columns $\vec{c}_1, \ldots, \vec{c}_v$ of C can be ordered so that there exist $m \in \mathbb{N}$ and $1 \le k_1 < \ldots < k_m = v$ such that

(1) $\sum_{i=1}^{k_1} \vec{c}_i = \vec{0}$ and,

(2) if $m > 1$, then for every $t \in \{2, \ldots, m\}$ we have $\alpha_{1,t}, \ldots, \alpha_{k_{m-1},t} \in R$ with $\sum_{i=k_{t-1}+1}^{k_t} \vec{c}_i = \sum_{i=1}^{k_{t-1}} \alpha_{i,t} \vec{c}_i$.

Denote by βX_d the Stone-Čech compactification of X with the discrete topology.

Definition 1.5. Let C be a $u \times v$ matrix and let X be a dense subsemigroup of $((0,1), \cdot)$. Then

$O_X = \{\mu \in \beta X_d : \text{for every } \varepsilon > 0,\ \mu((0,\varepsilon) \cap X) = 1\}$, and
$U_{X,C} = \{\mu \in O_X : \text{for all } \mu(A) = 1, \text{ there exist } x_1, \ldots, x_v \in A \text{ with } C\vec{x} = \vec{0}\}$.

At this point note that $\mathbb{Q}_p \cap (0,1)$ is a dense subsemigroup of $((0,1), \cdot)$. It is immediate that if αp^{-n}, $\beta p^{-m} \in \mathbb{Q}_p \cap (0,1)$, then $\alpha p^{-n} \cdot \beta p^{-m} = \alpha\beta p^{-(n+m)} \in \mathbb{Q}_p \cap (0,1)$, since $\alpha\beta < p^{n+m}$. Also, given $\varepsilon > 0$, take $p^{-n} < \varepsilon/2$. Then, every interval of length ε contains at least one point of the form αp^{-n}, with $0 < \alpha < p^n$ and we have the conclusion.

Lemma 1.6. *Let C be a $u \times v$ matrix with rational entries satisfying the columns condition over $\mathbb{Q}$ and $X = \mathbb{Q}_p \cap (0,1)$. Then $U_{X,C}$ is a two-sided ideal of $(O_X, \cdot)$ and a subsemigroup of $(O_X, +)$.*

Proof. The proof is analogous to Lemma 3.7 in [BHL]. The only thing we need to show, is that if $V = \{\mu \in \beta X_d : \text{for all } \mu(A) = 1, \text{ there exist } x_1, \ldots, x_v \in A \text{ such that } C\vec{x} = \vec{0}\}$, then $V \neq \emptyset$. It suffices to show that if X is partitioned into finitely many cells, one of then contains $x_1, \ldots, x_v$ with $C\vec{x} = \vec{0}$. C satisfies the column condition over $\mathbb{Q}$, so, by compactness, given $r \in \mathbb{N}$, there exists $n(r) \in \mathbb{N}$ such that whenever $\{1, \ldots, n(r)\}$ is partitioned into r sets, there exists a solution to $C\vec{x} = \vec{0}$ in one cell of the partition (see

[BHL]). So, by picking k with $p^k > n(r)$, whenever $\{\frac{1}{p^k}, \ldots, \frac{n(r)}{p^k}\} \subseteq X$ is partitioned into r sets, there exists a solution to $C\vec{x} = \vec{0}$ in one cell of the partition. $\square$

We now give a definition concerning the set $\mathbb{Q}_p$, analogous to Definition 5.2 of [BHL].

Definition 1.7. Fix an enumeration $\{C_n\}_{n\in\mathbb{N}}$ of the matrices with rational coefficients that satisfy the columns condition over $\mathbb{Q}$ so that $\{C_{2n}\}_{n\in\mathbb{N}}$ enumerates the matrices with integer entries that satisfy the columns condition over $\mathbb{Z}$. For each $n \in \mathbb{N}$, pick $(u(n), v(n)) \in \mathbb{N} \times \mathbb{N}$ such that C_n is a $u(n) \times v(n)$ matrix. For each $n \in \mathbb{N}$, let

(1) $\mathcal{R}_n = \big\{\{x_1, \ldots, x_{v(n)}\} \subseteq \mathbb{Q}_p : \ C_n\vec{x} = \vec{0}\big\}$, and

(2) $\mathcal{S}_n = \big\{\{x_1, \ldots, x_{v(2n)}\} \subseteq \mathbb{Q}_p$: for each $i \in \{1, 2, \ldots, u(2n)\}$, $\prod_{j=1}^{v(2n)} x_j^{d_{ij}} = 1\big\}$, where $C_{2n} =< d_{ij} >$.

Definition 1.8. Let $\{G_n\}_{n\in\mathbb{N}}$ be a sequence of finite subsets of $(0, 1)$. Set

$FS(\{G_n\}_{n\in\mathbb{N}}) = \{\sum_{n\in F} x_n : \ F \in [\mathbb{N}]^{<\omega}_{>0}$ and $x_n \in G_n$ for every $n \in \mathbb{N}\}$, and

$FP(\{G_n\}_{n\in\mathbb{N}}) = \{\prod_{n\in F} x_n : \ F \in [\mathbb{N}]^{<\omega}_{>0}$ and $x_n \in G_n$ for every $n \in \mathbb{N}\}$.

We are now in position to state the following result, the proof of which is analogous to Theorem 5.6 in [BHL]:

Theorem 1.9. *Let $\mathbb{Z}(p^\infty)^* = A_1 \cup \ldots \cup A_r$, $r \in \mathbb{N}$. Then, there exists $1 \le i_0 \le r$ and for each $n \in \mathbb{N}$ there exist choices of $G_n \in \mathcal{R}_n$, $H_{2n} \in \mathcal{S}_n$ and $H_{2n-1} \in \mathcal{R}_n$ with*

$$FS(\{G_n\}_{n\in\mathbb{N}}) \cup FP(\{H_n\}_{n\in\mathbb{N}}) \subseteq A_{i_0}.$$

An immediate implication of Theorem 1.9 is the following:

Corollary 1.10. *Let $\mathbb{Z}(p^\infty)^* = A_1 \cup \ldots \cup A_r$, $r \in \mathbb{N}$. Then, there exist $1 \le i_0 \le r$ and sequences $(x_n)_{n\in\mathbb{N}}$, $(y_n)_{n\in\mathbb{N}} \subseteq \mathbb{Z}(p^\infty)$ such that*

$$FS((x_n)_{n\in\mathbb{N}}) \cup FP((y_n)_{n\in\mathbb{N}}) \subseteq A_{i_0}.$$

Bergelson, Hindman and Leader also showed that, for partitions of $\mathbb{Q}_2$, one cannot quarantee to find a sequence $(x_n)_{n\in\mathbb{N}}$ with $FS((x_n)_{n\in\mathbb{N}}) \cup FP((x_n)_{n\in\mathbb{N}})$ contained in one cell. In fact, they showed that one cannot even quarantee to find a sequence $(x_n)_{n\in\mathbb{N}}$ with $FS((x_n)_{n\in\mathbb{N}}) \cup PP((x_n)_{n\in\mathbb{N}})$ monochromatic. With the same arguments and terminology (Theorem 5.9 in [BHL]) we have the following:

Theorem 1.11. *There exists a finite partition $\mathbb{Z}(p^\infty)^* = A_1 \cup \ldots \cup A_r$, such that there do not exist $1 \le i_0 \le r$ and a sequence $(x_n)_{n\in\mathbb{N}} \subseteq \mathbb{Z}(p^\infty)$, with*

$$FS((x_n)_{n\in\mathbb{N}}) \cup PP((x_n)_{n\in\mathbb{N}}) \subseteq A_{i_0}.$$

Proof. For $q = \sum_{t \in dom(q)} d_t p^{-t} \in \mathbb{Z}(p^\infty)^*$ the *start* of q is $s = \min dom(q)$ and the *end* the number $e = \max dom(q)$. If $|dom(q)| \geq 2$ (equivalently if q is not of the form $d_t p^{-t}$ with $t \in \mathbb{N}$, $1 \leq d_t \leq p-1$), then we say that the *type* t of q is 1 if $e-1 \in dom(q)$ and 0 if $e-1 \notin dom(q)$. A *previous point* y of a q with $|dom(q)| \geq 2$, is $\max\{1 \leq i \leq e-1 : i \in dom(q)\}$ if q is of type 0 and $\max\{1 \leq i \leq e-1 : \ i \notin dom(q)\}$ if q is of type 1. The *gap length* of q is $g = e - y$, so $g \geq 2$. If $|dom(q)| \geq 2$ then the *ratio* ρ of q is 1 if $g > s$ and 0 if $g \leq s$. We colorize the set $\mathbb{Z}(p^\infty)^*$ with 9 colors:

$$c(q) = \begin{cases} (t, g(mod 2), \rho), & \text{if } |dom(q)| \geq 2 \\ 0, & \text{if } |dom(q)| = 1 \end{cases}$$

We suppose that there exists a sequence $(x_n)_{n\in\mathbb{N}} \subseteq \mathbb{Z}(p^\infty)$, with $FS((x_n)_{n\in\mathbb{N}}) \cup PP((x_n)_{n\in\mathbb{N}})$ monochromatic. Noting that in general the $|dom(x_n)| \geq 2$ (since in general $dom(x_{n_1} + x_{n_2}) \neq 1$ for two elements with $dom(x_{n_1}) = dom(x_{n_2}) = 1$) and that $x_n \to 0$ as $n \to \infty$ (since the finite sums of x_n belong to $(0,1)$), we have a contradiction as in [BHL], Theorem 5.9. □

2. Density results for p-quasicyclic groups

The purpose of this section is to prove density results for p-quasicyclic groups giving a sufficient condition in order a subset of $\mathbb{Z}(p^\infty)$ to contain arbitrary long arithmetic progressions.

In 1927, van der Waerden proved (in [vdW]) that for any finite partition of the set of natural numbers, there exists a cell of the partition which contains arbitrary long arithmetic progressions, which is a (perhaps the most) fundamental result of Ramsey theory. The density version of the van der Waerden theorem, that any set of positive upper density in $\mathbb{N}$ possesses arbitrary long arithmetic progressions (the *upper density* of a subset $A \subseteq \mathbb{N}$ is defined by $\overline{d}(A) = \limsup_n \frac{|A \cap \{1,\ldots,n\}|}{n}$) was conjectured by Erdös and Turán in 1930's and established by Szemerédi in 1975 ([Sz]).

In order to obtain analogous results to Szemerédi's theorem for a semigroup $(S, +)$, we need to define a notion of "density" in S, such that every set with positive (upper) "density" in S to be combinatorial rich. In this paragraph, we will prove Szemerédi type theorems via left Følner sequences in $[\mathbb{Z}(p^\infty)]^{<\omega}_{>0}$.

Furstenberg, in 1977, introduced a correspondence principle which provides the link between density Ramsey theory and ergodic theory ([Fu]). Thus, Furstenberg reproved Szemerédi's result by recasting it as a multiple recurrence theorem in ergodic theory. Few years later, Furstenberg and Katznelson proved a recurrence theorem for commuting

IP-systems of measure preserving transformations in 1985 ([FuKa]), and a density Hales-Jewett-type theorem in 1991 ([FuKa2]), obtaining far reaching extensions of Szemerédi's theorem.

Definition 2.1. Let $T_1, \ldots, T_n, \ldots$ be a set of commuting transformations of a space. To every multi-index $\alpha = \{t_1, \ldots, t_l\} \in [\mathbb{N}]^{<\omega}_{>0},\ t_1 < \ldots < t_l$, we attach the transformation

$$\mathcal{T}_\alpha = T_{t_1} \ldots T_{t_l}.$$

The corresponding family $\{\mathcal{T}_\alpha\}_{\alpha \in [\mathbb{N}]^{<\omega}_{>0}}$ is an *IP-system* (of transformations).

Two IP-systems $\{\mathcal{T}^{(1)}_\alpha\}_{\alpha \in [\mathbb{N}]^{<\omega}_{>0}}$, $\{\mathcal{T}^{(2)}_\alpha\}_{\alpha \in [\mathbb{N}]^{<\omega}_{>0}}$ defined by $\{T^{(1)}_n\}_{n \in \mathbb{N}}$ and $\{T^{(2)}_n\}_{n \in \mathbb{N}}$ respectively, are *commuting* if $T^{(1)}_n T^{(2)}_m = T^{(2)}_m T^{(1)}_n$ for every n, $m \in \mathbb{N}$.

The following ergodic theoretical result is due to Furstenberg and Katznelson ([FuKa]):

Theorem 2.2. *Let $\{\mathcal{T}^{(1)}_\alpha\}_{\alpha \in [\mathbb{N}]^{<\omega}_{>0}}, \ldots, \{\mathcal{T}^{(k)}_\alpha\}_{\alpha \in [\mathbb{N}]^{<\omega}_{>0}}$ be k commuting IP-systems defined by the measure preserving transformations $\{T^{(j)}_n\}_{n \in \mathbb{N}}$, $1 \leq j \leq k$ of a measure space $(X, \mathcal{B}, \mu)$ with $\mu(X) = 1$ (i.e. $T^{(j)}_n$ is $\mathcal{B}$-measurable with $\mu(T^{(j)-1}_n(A)) = \mu(A)$ for every $A \in \mathcal{B}$, $1 \leq j \leq k$, $n \in \mathbb{N}$). If $A \in \mathcal{B}$ with $\mu(A) > 0$, then there exists an index α with*

$$\mu(A \cap \mathcal{T}^{(1)-1}_\alpha(A) \cap \ldots \cap \mathcal{T}^{(k)-1}_\alpha(A)) > 0.$$

Definition 2.3. Let $T_1, \ldots, T_n, \ldots$ be a set of commuting transformations of a space X. For every non-zero element $q \in \mathbb{Z}(p^\infty)$ with $dom(q) = \{t_1 < \ldots < t_l\}$, we define

$$\mathcal{T}^q(x) = T^{d_{t_1}}_{t_1} \ldots T^{d_{t_l}}_{t_l}(x) \text{ and } \mathcal{T}^0(x) = x \text{ for every } x \in X.$$

The corresponding family $\mathcal{T}$=$\{\mathcal{T}_q\}_{q \in \mathbb{Z}(p^\infty)}$ is an $\mathbb{Z}(p^\infty)$-*system* (of transformations).

Two $\mathbb{Z}(p^\infty)$-systems $\mathcal{T}_1$, $\mathcal{T}_2$ defined by $\{T_{n,1}\}_{n \in \mathbb{N}}$ and $\{T_{n,2}\}_{n \in \mathbb{N}}$ respectively, are *commuting* if $T_{n,1}T_{m,2} = T_{m,2}T_{n,1}$ for every n, $m \in \mathbb{N}$.

Remark 2.4. For a $\mathbb{Z}(p^\infty)$-system $\mathcal{T} = (\mathcal{T}^q)_{q \in \mathbb{Z}(p^\infty)}$, we have in general $\mathcal{T}^{p+q} \neq \mathcal{T}^p \mathcal{T}^q$ but if $p, q \in \mathbb{Z}(p^\infty)^*$ with $q \prec p$, then $\mathcal{T}^{p+q} = \mathcal{T}^p \mathcal{T}^q$.

Using Theorem 2.2, we can take the following:

Theorem 2.5. *Let $\mathcal{T}_1, \ldots, \mathcal{T}_k$ be k commuting $\mathbb{Z}(p^\infty)$-systems of measure preserving transformations of a measure space $(X, \mathcal{B}, \mu)$ with $\mu(X) = 1$. If $A \in \mathcal{B}$ with $\mu(A) > 0$, then there exists $q \in \mathbb{Z}(p^\infty)^*$ with*

$$\mu(A \cap (\mathcal{T}^q_1)^{-1}(A) \cap \ldots \cap (\mathcal{T}^q_k)^{-1}(A)) > 0.$$

Proof. Let $A \in \mathcal{B}$ with $\mu(A) > 0$. Suppose that $\mathcal{T}_i$ is defined from the commuting measure preserving transformations $\{T_{n,i}\}_{n \in \mathbb{N}}$ of X. For every $n \in \mathbb{N}$ choose a natural number

$1 \leq d_n \leq p-1$. For $n \in \mathbb{N}$, $1 \leq i \leq k$, set $\phi_n^{(i)} = T_{n,i}^{d_n}$ and let the corresponding IP-systems $\{\phi_\alpha^{(i)}\}_{\alpha \in [\mathbb{N}]_{>0}^{<\omega}}$.

According to Theorem 2.2 there exists $\alpha = \{t_1 < \ldots < t_l\} \in [\mathbb{N}]_{>0}^{<\omega}$ with

$$\mu(A \cap \phi_\alpha^{(1)-1}(A) \cap \ldots \cap \phi_\alpha^{(k)-1}(A)) > 0.$$

Since $\phi_\alpha^{(i)} = \mathcal{T}_i^q$ for all $1 \leq i \leq k$, where

$$q = \sum_{t \in \alpha} d_t p^{-t} \in \mathbb{Z}(p^\infty)^*,$$

we have that $\mu(A \cap (\mathcal{T}_1^q)^{-1}(A) \cap \ldots \cap (\mathcal{T}_k^q)^{-1}(A)) > 0$. □

Left Følner sequences. If $(S, +)$ is an infinite countable left cancellative semigroup (i.e. $a+b = a+c \Rightarrow b = c$ for every $a, b, c \in S$), then we can find a left Følner sequence in $[S]_{>0}^{<\omega}$.

Definition 2.6. Let $(S, +)$ be a semigroup. A *left Følner sequence* in $[S]_{>0}^{<\omega}$ is a sequence $\{F_n\}_{n \in \mathbb{N}}$ in $[S]_{>0}^{<\omega}$ such that for each $s \in S$,

$$\lim_{n \to \infty} \frac{|(s + F_n) \triangle F_n|}{|F_n|} = 0,$$

where $A \triangle B = (A \setminus B) \cup (B \setminus A)$.

Given a left Følner sequence $\mathcal{F} = \{F_n\}_{n \in \mathbb{N}}$ in $[S]_{>0}^{<\omega}$, there is a natural notion of upper density associated with $\mathcal{F}$, namely

$$\overline{d}_{\mathcal{F}}(A) = \limsup_{n \to \infty} \frac{|A \cap F_n|}{|F_n|}.$$

For $A \subseteq S$ and $t \in S$ we set $-t + A = \{s \in S : t + s \in A\}$. Hindman and Strauss in [HS] proved the following result concerning left cancellative semigroups:

Theorem 2.7. *Let S be an infinite countable left cancellative semigroup, let $\mathcal{F} = \{F_n\}_{n \in \mathbb{N}}$ be a left Følner sequence in $[S]_{>0}^{<\omega}$, and let $A \subseteq S$. There is a countably additive measure μ on the set $\mathcal{B}$ of Borel subsets of βS such that*

(1) $\mu(\overline{A}) = \overline{d}_{\mathcal{F}}(A)$,

(2) *for all $B \subseteq S$, $\mu(\overline{B}) \leq \overline{d}_{\mathcal{F}}(B)$,*

(3) *for all $B \in \mathcal{B}$ and all $t \in S$, $\mu(-t + B) = \mu(B) = \mu(t + B)$, and*

(4) $\mu(\beta S) = 1$.

Using Theorem 2.7 we will state and prove a Szemerédi-type theorem for the group $\mathbb{Z}(p^\infty)$ (the proof is analogous to Theorem 5.6 in [HS]).

Theorem 2.8. *Let $\mathcal{F} = \{F_n\}_{n\in\mathbb{N}}$ be a left Følner sequence in $[\mathbb{Z}(p^\infty)]^{<\omega}_{>0}$, and let $A \subseteq \mathbb{Z}(p^\infty)$ such that $\overline{d}_{\mathcal{F}}(A) > 0$. Then for each $k \in \mathbb{N}$ there exists $q \in \mathbb{Z}(p^\infty)^*$ such that*

$$\overline{d}_{\mathcal{F}}(A \cap (-q + A) \cap \ldots \cap (-kq + A)) > 0.$$

Proof. Let $\mathcal{B}$ be the set of Borel subsets of $\beta\mathbb{Z}(p^\infty)$. Pick a countably additive measure μ on $\mathcal{B}$ which satisfies the conditions of Theorem 2.7. Let $k \in \mathbb{N}$. For $l \in \{1, \ldots, k\}$ and $\nu \in \beta\mathbb{Z}(p^\infty)$ let $T_{l,m}(\nu) = \mu_{lp^{-m}} + \nu$ for $m \in \mathbb{N}$. Each $T_{l,n},\ n \in \mathbb{N}$ is continuous, so if $\mathcal{T}_l$ is defined from $\{T_{l,n}\}_{n\in\mathbb{N}}$ we have that $(\mathcal{T}_l^q)^{-1}(B) \in \mathcal{B}$ for every $B \in \mathcal{B}$, $q \in \mathbb{Z}(p^\infty)$, $1 \leq l \leq k$ and the $\mathcal{T}_l$, $1 \leq l \leq k$ are commuting $\mathbb{Z}(p^\infty)$-systems of measure preserving transformations on the measure space $(\beta\mathbb{Z}(p^\infty), \mathcal{B}, \mu)$. $\mu(\overline{A}) = \overline{d}_{\mathcal{F}}(A) > 0$, so, according to Theorem 2.5 there exists $q \in \mathbb{Z}(p^\infty)^*$ such that $\mu(\overline{A} \cap (\mathcal{T}_1^q)^{-1}(\overline{A}) \cap \ldots \cap (\mathcal{T}_k^q)^{-1}(\overline{A})) > 0$. Then, we have

$$\begin{aligned}\overline{d}_{\mathcal{F}}(A \cap (-q + A) \cap \ldots \cap (-kq + A)) = \mu(\overline{A \cap (-q + A) \cap \ldots \cap (-kq + A)}) = \\ = \mu(\overline{A} \cap (\mathcal{T}_1^q)^{-1}(\overline{A}) \cap \ldots \cap (\mathcal{T}_k^q)^{-1}(\overline{A})) > 0,\end{aligned}$$

which is the required condition. □

Corollary 2.9. *Let $\mathcal{F} = \{F_n\}_{n\in\mathbb{N}}$ be a left Følner sequence in $[\mathbb{Z}(p^\infty)]^{<\omega}_{>0}$, and let $A \subseteq \mathbb{Z}(p^\infty)$ such that $\overline{d}_{\mathcal{F}}(A) > 0$. Then for each $k \in \mathbb{N}$ there exist $q \in \mathbb{Z}(p^\infty)^*$ and $r \in A$ such that*

$$r + jq \in A \text{ for every } 0 \leq j \leq k.$$

Proof. Let $k \in \mathbb{N}$. According to the proof of Theorem 2.8 there exists $q \in \mathbb{Z}(p^\infty)^*$ such that $\overline{d}_{\mathcal{F}}(A \cap (-q + A) \cap \ldots \cap (-kq + A)) > 0$. Pick $r \in A \cap (-q + A) \cap \ldots \cap (-kq + A)$. □

We remark that, defining for a left Følner sequence in $[\mathbb{Z}(p^\infty)]^{<\omega}_{>0}$, $\mathcal{F} = \{F_n\}_{n\in\mathbb{N}}$, and $A \subseteq \mathbb{Z}(p^\infty)$ the density

$$d^*_{\mathcal{F}}(A) = \sup\{\alpha :\ (\forall m \in \mathbb{N})(\exists n \geq m)(\exists q \in \mathbb{Z}(p^\infty))(|A \cap (q + F_n)| \geq \alpha|F_n|)\}$$

we have the conclusions of Theorem 2.8 and Corollary 2.9 replacing $\overline{d}_{\mathcal{F}}$ with $d^*_{\mathcal{F}}$ (using Theorem 4.6 from [HS], which is the analogous of Theorem 2.7 for $d^*_{\mathcal{F}}$).

We will now prove a finite form of a Hales-Jewett type theorem for the group $\mathbb{Z}(p^\infty)$.

Definition 2.10. Let $\Sigma = \{\alpha_1, \ldots, \alpha_k\},\ k \in \mathbb{N}$ a finite set and $\upsilon \notin \Sigma$. A *combinatorial line* in $L(\Sigma)$ is the set of words obtained by a variable word $w(\upsilon)$ by substituting for the variable υ all the symbols $\alpha_1, \ldots, \alpha_k$. We also denote by $L_n(\Sigma)$ the subset of $L(\Sigma)$ consisting of all the words of length n.

Furstenberg and Katznelson in [FuKa2] shown the following theorem:

Theorem 2.11. *Let $\Sigma = \{\alpha_1, \dots, \alpha_k\}$, $k \in \mathbb{N}$ a finite alphabet. If $A \subseteq L(\Sigma)$ and $\limsup_n \frac{|A \cap L_n(\Sigma)|}{k^n} > 0$, then A contains a combinatorial line.*

For every $n \in \mathbb{N}$ choose distinct natural numbers $t_1^n, \dots, t_n^n$ and form the set

$$\mathbb{Z}(p^\infty)(t_1^n, \dots, t_n^n) = \{\textstyle\sum_{j=1}^n d_{t_j^n} p^{-t_j^n} : 0 \leq d_{t_j^n} \leq p-1\} \subseteq \mathbb{Z}(p^\infty).$$

We say that $(\mathbb{Z}(p^\infty)(t_1^n, \dots, t_n^n))_{n\in\mathbb{N}}$ is a p-sequence.

For $\Sigma = \{0, 1, \dots, p-1\}$ we define $g : L(\Sigma) \to \cup_{n\in\mathbb{N}} \mathbb{Z}(p^\infty)(t_1^n, \dots, t_n^n)$ with

$$g(w_1 \dots w_n) = \textstyle\sum_{j=1}^n w_j p^{-t_j^n}.$$

Note that for every $n \in \mathbb{N}$, $g|_{L_n(\Sigma)} : L_n(\Sigma) \to \mathbb{Z}(p^\infty)(t_1^n, \dots, t_n^n)$ is onto.

Using Theorem 2.11 we have the following finite density Hales-Jewett type theorem:

Theorem 2.12. *Let p be a prime number and $(\mathbb{Z}(p^\infty)(t_1^n, \dots, t_n^n))_{n\in\mathbb{N}}$ be a p-sequence. If $A \subseteq \mathbb{Z}(p^\infty)$ with $\limsup_n \frac{|A \cap \mathbb{Z}(p^\infty)(t_1^n, \dots, t_n^n)|}{p^n} > 0$, then there exist $r \in A$ and $q \in \mathbb{Z}(p^\infty)^*$ such that $r + iq \in A$ for every $i = 0, 1, \dots, p-1$.*

Proof. Since $|g^{-1}(A) \cap L_n(\Sigma)| \geq |A \cap \mathbb{Z}(p^\infty)(t_1^n, \dots, t_n^n)|$ for every $n \in \mathbb{N}$, according to Theorem 2.11, $g^{-1}(A)$ contains a combinatorial line $w(v)$, $v \notin \Sigma$. If $c(w) = \{t \in dom(w) : w_t \in \Sigma\}$ and $v(w) = \{t \in dom(w) : w_t = v\}$, then for $r = \sum_{t \in c(w)} w_t c_t$ and $q = \sum_{t \in v(w)} c_t$, where $c_t = p^{-t}$ we have that $g(w(i)) = r + iq \in A$ for every $0 \leq i \leq p-1$. $\square$

3. Simple $\mathbb{Z}(p^\infty)$-dynamical systems

We introduce the notion of a simple $\mathbb{Z}(p^\infty)$-dynamical system defined from a compact metric space X and a sequence $\{T_n\}_{n\in\mathbb{N}}$ of continuous functions from X to itself (Definition 3.1). We prove a recurrence theorem for such systems in Theorem 3.2, analogous to Theorem 0.2, in case $l = 1$, of Furstenberg and Weiss. Theorem 3.2 is an implication of a strengthened partition theorem for the group $\mathbb{Z}(p^\infty)$ (Theorem 1.2). The inverse implication is partially correct since Theorem 3.2 implies a weaker form of Theorem 1.2.

Definition 3.1. Let X be a compact metric space and $\mathcal{T}$ a $\mathbb{Z}(p^\infty)$-system defined by a family of commuting continuous maps (see Definition 2.3). We say that $(X, \mathcal{T})$ is a *simple $\mathbb{Z}(p^\infty)$-dynamical system.*

Using Theorem 1.2 we can prove a recurrence result for simple $\mathbb{Z}(p^\infty)$-dynamical systems analogous to the result of Furstenberg and Weiss (Theorem 0.2, case $l = 1$).

Theorem 3.2. *Let $(X, \mathcal{T})$ be a simple $\mathbb{Z}(p^\infty)$-dynamical system and $k \in \mathbb{N}$. Then, for every $\varepsilon > 0$ there exist $\beta \in \mathbb{Z}(p^\infty)^*$ with $k < \min dom(\beta)$ and $x_0 \in X$ such that*

$$\mathcal{T}^{i\beta}(x_0) \in B(x_0, \varepsilon) \text{ for every } 0 \leq i \leq p-1.$$

Proof. Let $k \in \mathbb{N}$ and $\varepsilon > 0$. Since X is compact, we have $X = \bigcup_{i=1}^m B(x_i, \frac{\varepsilon}{2})$ for some $x_1, \ldots, x_m \in X,\ m \in \mathbb{N}$. Let $x \in X$. We form a partition of the set $\mathbb{Z}(p^\infty)$: $\mathbb{Z}(p^\infty) = Z_1 \cup \ldots \cup Z_m$, where

$$q \in Z_i \ \Leftrightarrow\ \mathcal{T}^q(x) \in B(x_i, \tfrac{\varepsilon}{2}) \text{ and } \mathcal{T}^q(x) \notin B(x_j, \tfrac{\varepsilon}{2}) \text{ for } j < i.$$

According to Theorem 1.2, there exist $1 \leq i_0 \leq m$ and C, V non-empty sets with $C \cap V = \emptyset$, such that

$$p_i^* = \sum_{t \in C} d_t p^{-t} \ + \ i \sum_{t \in V} p^{-t} \in Z_{i_0}$$

for every $0 \leq i \leq p-1$, where $1 \leq d_t \leq p-1,\ t \in \mathbb{N}$ and $k < \min dom(p_1^*)$. Equivalently, if

$$\beta = \textstyle\sum_{t \in V} p^{-t} \text{ and } \gamma = \sum_{t \in C} d_t p^{-t}$$

we have that $\mathcal{T}^{\gamma\, +\, i\beta}(x) \in B(x_{i_0}, \frac{\varepsilon}{2})$ for every $1 \leq i \leq p-1$. Let $x_0 = \mathcal{T}^\gamma(x)$. Then $\mathcal{T}^{i\beta}(x_0) \in B(x_0, \varepsilon)$ for every $0 \leq i \leq p-1$. □

According to the previous proof, Theorem 3.2 is an implication of Theorem 1.2. We will show that the inverse implication is partially correct. So, Theorem 3.2 can be considered as a topological partition theorem for the group $\mathbb{Z}(p^\infty)$.

Theorem 3.3. *Let k be a real number. If $\mathbb{Z}(p^\infty) = Z_1 \cup \ldots \cup Z_r,\ r \in \mathbb{N}$, then there exists $1 \leq i_0 \leq r$ $\alpha \in \mathbb{Z}(p^\infty)$ and $\beta \in \mathbb{Z}(p^\infty)^*$ with $k < \min dom(\beta)$ such that $\alpha + i\beta \subseteq Q_{i_0}$ for every $0 \leq i \leq p-1$.*

Proof. Let $k \in \mathbb{N},\ \Omega = \{1, \ldots, r\}^{\mathbb{Z}(p^\infty)}$ and enumerate $\mathbb{Z}(p^\infty) = \bigcup_{n \in \mathbb{N}} \{q_n\}$ with $q_1 = 0$. Ω becomes compact metric space with metric

$$d(\omega, \omega') = \inf\{\tfrac{1}{t} :\ \omega(q_i) = \omega'(q_i) \text{ for } 1 \leq i < t\}.$$

Let $\mathcal{T}$ be a $\mathbb{Z}(p^\infty)$-system which is defined from $\{T_n\}_{n \in \mathbb{N}}$, where

$$T_n \omega(q) = \omega(q + p^{-n}),\ n \in \mathbb{N}.$$

Define a specific point $\omega \in \Omega$ according to the rule $\omega(q) = i \quad \Leftrightarrow \quad q \in Z_i$ and $q \notin Z_j$ for $j < i$ and let $X = \overline{\{\mathcal{T}^{s_1} \ldots \mathcal{T}^{s_m}\omega : \ s_i \in \mathbb{Z}(p^\infty), \ m \in \mathbb{N}, \ 1 \leq i \leq m\}}$. Then, $\mathcal{T}$ is a $\mathbb{Z}(p^\infty)$-systmem of X. According to Theorem 3.2 (for $\varepsilon = 1$) there exist $\beta \in \mathbb{Z}(p^\infty)^*$ with $k < \min dom(\beta)$ and $x_0 \in X$ such that $d(\mathcal{T}^{i\beta}(x_0), x_0) < 1$ for every $0 \leq i \leq p-1$. Then, we have

$$(*) \qquad x_0(0) = x_0(i\beta) \text{ for every } 0 \leq i \leq p-1.$$

$x_0 \in X$, so, there exist $s_1, \ldots, s_m \in \mathbb{Z}(p^\infty)$ such that x_0 is close to $\mathcal{T}^{s_1} \ldots \mathcal{T}^{s_m}\omega$. Set $\alpha = s_1 + \ldots + s_m \in \mathbb{Z}(p^\infty)$. According to $(*)$ we have that $\omega(\alpha) = \omega(\alpha + i\beta)$ for every $0 \leq i \leq p-1$, thus, we have $\alpha + i\beta \in Z_{\omega(\alpha)}$ for every $0 \leq i \leq p-1$. $\square$

Remark 3.4. Since in general $\mathcal{T}^{p+q} \neq \mathcal{T}^p\mathcal{T}^q$, we cannot prove Theorem 3.2 form Theorem 3.3 for we cannot have control of the support of $\alpha \in \mathbb{Z}(p^\infty)$ relating to the support of $\beta \in \mathbb{Z}(p^\infty)^*$.

4. Uniform $\mathbb{Z}(p^\infty)$-Recurrence and Minimal $\mathbb{Z}(p^\infty)$-Dynamical Systems

In the present paragraph we introduce the notion of a $\mathbb{Z}(p^\infty)$-dynamical system (Definition 4.1 below) and also the notion of a uniform $\mathbb{Z}(p^\infty)$-recurrent point of a such system (Definition 4.8). Defining the minimal $\mathbb{Z}(p^\infty)$-dynamical systems (Definition 4.3) we prove that every $\mathbb{Z}(p^\infty)$-dynamical system has uniform $\mathbb{Z}(p^\infty)$-recurrent points. In fact a $\mathbb{Z}(p^\infty)$-dynamical system is minimal if and only if every point of the system is uniform $\mathbb{Z}(p^\infty)$-recurrent (Theorems 4.9 and 4.11).

Definition 4.1. Let X be a compact metric space and $\mathcal{T}_1, \ldots, \mathcal{T}_l$, be l commuting $\mathbb{Z}(p^\infty)$-systems defined by commuting homeomorphisms of X into itself. We say that $(X, \mathcal{T}_1, \ldots, \mathcal{T}_l)$ is a $\mathbb{Z}(p^\infty)$-*dynamical system.*

Remark 4.2. For a $\mathbb{Z}(p^\infty)$-system $\mathcal{T} = (\mathcal{T}^q)_{q \in \mathbb{Z}(p^\infty)}$ of a $\mathbb{Z}(p^\infty)$-dynamical system $(X, \mathcal{T})$ we have in general that $\mathcal{T}^{-q} \neq (\mathcal{T}^{-1})^q$, where we have set $(\mathcal{T}^{-1})^q = (\mathcal{T}^q)^{-1}$.

We will define and characterize the minimal $\mathbb{Z}(p^\infty)$-dynamical systems.

Definition 4.3. A $\mathbb{Z}(p^\infty)$-dynamical system $(X, \mathcal{T}_1, \ldots, \mathcal{T}_l)$, where $\mathcal{T}_i = (\mathcal{T}_i^q)_{q \in \mathbb{Z}(p^\infty)}$ is defined from the commuting homeomorphisms $\{T_{i,n}\}_{n \in \mathbb{N}}$ of X for $1 \leq i \leq l$, is said to be *minimal* if for every closed $Y \subseteq X$ with $T_{i,n}(Y) = Y$ for every $1 \leq i \leq l$, $n \in \mathbb{N}$ we have that $Y = X$ or $Y = \emptyset$.

Proposition 4.4. *Let $(X, \mathcal{T}_1, \ldots, \mathcal{T}_l)$ be a $\mathbb{Z}(p^\infty)$-dynamical system, where for $1 \leq i \leq l$, $\mathcal{T}_i = (\mathcal{T}_i^q)_{q \in \mathbb{Z}(p^\infty)}$ is defined from the commuting homeomorphisms $\{T_{i,n}\}_{n \in \mathbb{N}}$. Let G be the group of homeomorphisms of X generated by the functions $T_{i,n}$, for $n \in \mathbb{N}$ and $1 \leq i \leq l$. The following are equivalent:*

(i) *$(X, \mathcal{T}_1, \ldots, \mathcal{T}_l)$ is minimal.*

(ii) *$\overline{\{S(x) :\ S \in G\}} = X$ for every $x \in X$.*

(iii) *For every non-empty open set $V \subseteq X$ there exist a non-empty finite subset of G, F, such that $\bigcup_{S \in F} S^{-1}(V) = X$.*

Proof. $(i) \Rightarrow (ii)$ Let $x \in X$. For every $n \in \mathbb{N}$, $1 \leq i \leq l$, we have that $T_{i,n}(\overline{\{S(x) :\ S \in G\}}) = \overline{\{S(x) :\ S \in G\}}$. Since $\overline{\{S(x) :\ S \in G\}} \neq \emptyset$ and $(X, \mathcal{T}_1, \ldots, \mathcal{T}_l)$ is a minimal dynamical system, we have that $\overline{\{S(x) :\ S \in G\}} = X$.

$(ii) \Rightarrow (i)$ If Y is a closed non-empty closed subset of X with $T_{i,n}(Y) = Y$ for every $n \in \mathbb{N}$, $1 \leq i \leq l$, then $X = \overline{\{S(y) :\ S \in G\}} \subseteq Y$ for every $y \in Y$. Then $Y = X$, thus, $(X, \mathcal{T}_1, \ldots, \mathcal{T}_l)$ is minimal.

$(i) \Rightarrow (iii)$ For every non-empty open set V we have $\bigcup_{S \in G} S^{-1}(V) = X$. From the compactness of X we have the conclusion.

$(iii) \Rightarrow (i)$ Let $(X, \mathcal{T}_1, \ldots, \mathcal{T}_l)$ is not minimal. Let Y be a non-empty closed invariant proper subset of X and $V = X \setminus Y$. Then $\bigcup_{S \in G} S^{-1}(V) \neq X$, a contradiction. $\square$

Definition 4.5. Let $(X, \mathcal{T}_1, \ldots, \mathcal{T}_s)$ be a $\mathbb{Z}(p^\infty)$-dynamical system and $Y \subseteq X$. We say that the system $(Y, \mathcal{T}_1|_Y, \ldots, \mathcal{T}_s|_Y)$ is a *subsystem* of $(X, \mathcal{T}_1, \ldots, \mathcal{T}_s)$ if

(i) Y is a closed subset of X, and

(ii) $T_{i,n}(Y) = Y$ for every $1 \leq i \leq s,\ n \in \mathbb{N}$.

Proposition 4.6. *Every $\mathbb{Z}(p^\infty)$-dynamical system has a minimal subsystem.*

Proof. Let $(X, \mathcal{T}_1, \ldots, \mathcal{T}_l)$ be a $\mathbb{Z}(p^\infty)$-dynamical system, where for $1 \leq i \leq l$, $\mathcal{T}_i = (\mathcal{T}_i^q)_{q \in \mathbb{Z}(p^\infty)}$ is defined from the commuting homeomorphisms $\{T_{i,n}\}_{n \in \mathbb{N}}$. Let $\mathcal{L} = \{Y \subseteq X :\ Y \neq \emptyset,\ Y \text{ closed and } T_{i,n}(Y) = Y \text{ for every } n \in \mathbb{N},\ 1 \leq i \leq l\}$. $\mathcal{L} \neq \emptyset$ since $X \in \mathcal{L}$. Let $\mathcal{D} \subseteq \mathcal{L}$ be a family totaly ordered by inclusion. $\mathcal{D}$ has the finite intersection property and since X is compact we have that $A := \bigcap_{Y \in \mathcal{D}} Y \neq \emptyset$, with $A \subseteq Y$ for every $Y \in \mathcal{D}$. According to Zorn's lemma there exists a minimal $Y_0 \in \mathcal{L}$. Then, $(Y_0, \mathcal{T}_1|_{Y_0}, \ldots, \mathcal{T}_l|_{Y_0})$ is a minimal subsystem. $\square$

We will introduce the notion of uniformly $\mathbb{Z}(p^\infty)$-recurrent points for a $\mathbb{Z}(p^\infty)$-dynamical system. Firstly we will recall the notion of a syndetic subset of an abelian (semi-)group.

Definition 4.7. A subset E of an abelian (semi)group G is *syndetic* if there exists $F \in [G]^{<\omega}_{>0}$ such that $G = \bigcup_{g \in F} \{s \in G : \ g + s \in E\}$.

Definition 4.8. Let $(X, \mathcal{T}_1, \ldots, \mathcal{T}_l)$ be a $\mathbb{Z}(p^\infty)$-dynamical system, where for $1 \leq i \leq l$, $\mathcal{T}_i$ is defined from the commuting homeomorphisms $\{T_{i,n}\}_{n \in \mathbb{N}}$. A point $x \in X$ is *uniformly $\mathbb{Z}(p^\infty)$-recurrent* for $(X, \mathcal{T}_1, \ldots, \mathcal{T}_l)$ if for any neighborhood V of x, the set $\{S \in G : S(x) \in V\}$ is syndetic, where G is the group of homeomorphisms of X generated by the functions $T_{i,n}$ for every $n \in \mathbb{N}$ and $1 \leq i \leq l$.

The following theorem gives the connection between minimal $\mathbb{Z}(p^\infty)$-dynamical systems and uniformly $\mathbb{Z}(p^\infty)$-recurrent points.

Theorem 4.9. *If $(X, \mathcal{T}_1, \ldots, \mathcal{T}_l)$ is a minimal $\mathbb{Z}(p^\infty)$-dynamical system, then every point $x \in X$ is uniformly $\mathbb{Z}(p^\infty)$-recurrent.*

Proof. If $\mathcal{T}_i$, for $1 \leq i \leq l$, is defined from the commuting homeomorphisms $\{T_{i,n}\}_{n \in \mathbb{N}}$ and G is the group of homeomorphisms of X generated by the functions $T_{i,n}$ for every $n \in \mathbb{N}$ and $1 \leq i \leq l$, then for every $x \in X$ and every non-empty open set $V \subseteq X$, the set $\{S \in G : \ S(x) \in V\}$ is syndetic. Indeed, according to Proposition 4.4 we have that $\bigcup_{i=1}^m S_i^{-1}(V) = X$ for some $m \in \mathbb{N}$, $S_1, \ldots, S_m \in G$. So, for every $S \in G$ we have $S_i(S(x)) \in V$ for some $1 \leq i \leq m$, or $S_i S \in \{S \in G : \ S(x) \in V\}$. □

Corollary 4.10. *For any $\mathbb{Z}(p^\infty)$-dynamical system $(X, \mathcal{T}_1, \ldots, \mathcal{T}_l)$, the set of uniformly $\mathbb{Z}(p^\infty)$-recurrent points is non-empty.*

Proof. Immediate from Proposition 4.6 and Theorem 4.9. □

Now we can characterize the minimal subsystems of a $\mathbb{Z}(p^\infty)$-dynamical system via the uniformly $\mathbb{Z}(p^\infty)$-recurrent points of the system.

Theorem 4.11. *Let $(X, \mathcal{T}_1, \ldots, \mathcal{T}_l)$ be a $\mathbb{Z}(p^\infty)$-dynamical system, where for $1 \leq i \leq l$, $\mathcal{T}_i$ is defined from the commuting homeomorphisms $\{T_{i,n}\}_{n \in \mathbb{N}}$ and G be the group of homeomorphisms of X generated by the functions $T_{i,n}$ for every $n \in \mathbb{N}$ and $1 \leq i \leq l$. Then $(Y, \mathcal{T}_1|_Y, \ldots, \mathcal{T}_l|_Y)$ is a minimal subsystem of $(X, \mathcal{T}_1, \ldots, \mathcal{T}_l)$ if and only if $Y = \overline{\{S(x) : \ S \in G\}}$ for x a uniformly $\mathbb{Z}(p^\infty)$-recurrent point of $(X, \mathcal{T}_1, \ldots, \mathcal{T}_l)$.*

Proof. It suffices to prove that if $y \in \overline{\{S(x) : \ S \in G\}}$ then x belongs to $\overline{\{S(y) : \ S \in G\}}$. Assume otherwise and let V be an open neighborhood of x with $\overline{V} \cap \overline{\{S(y) : \ S \in G\}} = \emptyset$. x is a uniformly $\mathbb{Z}(p^\infty)$-recurrent point, so there exists a finite set $\{S_1, \ldots, S_m\}$, $m \in \mathbb{N}$

of elements of G such that for every $S \in G$ to have $S_i(S(x)) \in V$ for some $1 \leq i \leq m$. So, $\{S(x) : \ S \in G\} \subseteq \bigcup_{i=1}^m S_i^{-1}(V)$, thus $y \in \overline{\{S(x) : \ S \in G\}} \subseteq \bigcup_{i=1}^m S_i^{-1}(\overline{V})$. Then we have $\overline{V} \cap \overline{\{S(y) : \ S \in G\}} \neq \emptyset$, a contradiction. $\square$

5. The recurrence properties of $\mathbb{Z}(p^\infty)$-dynamical systems

In Theorems 5.1 and 5.2 below, we prove that $\mathbb{Z}(p^\infty)$-dynamical systems has significant recurrence properties, analogous to those of the classical dynamical systems. So, Theorem 5.1 is an analogous result of Theorem 0.2 of Furstenberg-Weiss to $\mathbb{Z}(p^\infty)$-dynamical systems and Theorem 5.2 is an equivalent reformulation of Theorem 5.1 (for the analogous results for systems indexed by $\mathbb{N}$ or $\mathbb{Z}$ see [Fu], [FuW], [M] and for systems indexed by $\mathbb{Q}$ see [K]).

As a consequence of Theorem 5.1, which is a multiple version of Theorem 3.2 in case the transformations T_i are invertible, we get a combinatorial result for the group $\mathbb{Z}(p^\infty)$ (Theorem 5.3), proving that for $l \in \mathbb{N}$ and any finite partition of $\mathbb{Z}(p^\infty)^l$, one of the cells of the partition contains (generalized) affine images of every finite subset of $\mathbb{Z}(p^\infty)^l$. We also remark that syndetic subsets of $\mathbb{Z}(p^\infty)^l$ have the same property.

Theorem 5.1. *Let $l \in \mathbb{N}$ and $(X, \mathcal{T}_1, \ldots, \mathcal{T}_l)$ a $\mathbb{Z}(p^\infty)$-dynamical system and $k \in \mathbb{N}$. For every $\varepsilon > 0$ there exists $\beta \in \mathbb{Z}(p^\infty)^*$ with $k < \min dom(\beta)$ and $x_0 \in X$ such that*

$$\mathcal{T}_j^{i\beta}(x_0) \in B(x_0, \varepsilon) \text{ for every } 0 \leq i \leq p-1,\ 1 \leq j \leq l.$$

Theorem 5.2. *Let $l \in \mathbb{N}$ and $(X, \mathcal{T}_1, \ldots, \mathcal{T}_l, R)$ a minimal $\mathbb{Z}(p^\infty)$-dynamical system and $k \in \mathbb{N}$. Then for every non-empty open set U there exists $\beta \in \mathbb{Z}^*$ with $k < \min dom(\beta)$ such that*

$$\bigcap_{0 \leq i \leq p-1} (U \cap (\mathcal{T}_1^{i\beta})^{-1}U \cap \ldots \cap (\mathcal{T}_l^{i\beta})^{-1}U) \neq \emptyset.$$

Proof. Our method of proof is induction on l and consists of three steps:

(1) We will show that Theorem 5.1 holds for $l = 1$,

(2) if Theorem 5.1 holds for some $l \in \mathbb{N}$ then Theorem 5.2 also holds for l, and

(3) if Theorem 5.2 holds for some $l \in \mathbb{N}$ then Theorem 5.1 holds for $l + 1$.

For $l = 1$ we have the conclusion of Theorem 5.1 from Theorem 3.2.

Let $l \in \mathbb{N}$ such that Theorem 5.1 holds. Let $(X, \mathcal{T}_1, \ldots, \mathcal{T}_l, R)$ a minimal $\mathbb{Z}(p^\infty)$-dynamical system, where $\mathcal{T}_i$ is defined from the commuting homeomorphisms $\{T_{i,n}\}_{n \in \mathbb{N}}$ of X for $1 \leq i \leq l$, R is defined from the commuting homeomorphisms $\{R_n\}_{n \in \mathbb{N}}$ of X and $k \in \mathbb{N}$. Let $U \subseteq X$ a non-empty open set. There exists $u \in U$ and $\varepsilon > 0$ such that

$B(u,\varepsilon) \subseteq U$. Let $V = B(u, \frac{\varepsilon}{2}) \subseteq U$. Then, for every $x \in X$ with $d(x,V) := \inf\{d(x,y) : y \in V\} < \frac{\varepsilon}{2}$ we have that $x \in U$. Let G be the group of homeomorphisms generated by $\{T_{i,n}\}_{n\in\mathbb{N}},\ 1 \leq i \leq l$ and $\{R_n\}_{n\in\mathbb{N}}$.

Since the system $(X, \mathcal{T}_1, \ldots, \mathcal{T}_l, R)$ is minimal there exists some $S_1, \ldots, S_m \in G,\ m \in \mathbb{N}$ with $X = \bigcup_{i=1}^m S_i^{-1}V$ $(*)$. Since X is compact, every $S_i,\ 1 \leq i \leq m$ is uniformly continuous, so there exists $\delta > 0$ such that if $y, z \in X$ with $d(y,z) < \delta$ then $d(S_i(y), S_i(z)) < \frac{\varepsilon}{2}$ for $1 \leq i \leq m$. According to Theorem 5.1 there exists $y \in X,\ \beta \in \mathbb{Z}(p^\infty)^*$ with $k < \min dom(\beta)$ such that $d(y, \mathcal{T}_j^{i\beta}(y)) < \delta$ for every $0 \leq i \leq p-1,\ 1 \leq j \leq l$. From $(*)$ we have that there exists $1 \leq i \leq m$ such that $y \in S_i^{-1}V$. Set $x = S_i(y) \in V$. Since S_j commutes with the $\{T_{i,n}\}_{n\in\mathbb{N}}$, for $1 \leq i \leq l$, we have that $d(x, \mathcal{T}_j^{i\beta}(x)) < \frac{\varepsilon}{2}$ for every $0 \leq i \leq p-1,\ 1 \leq j \leq l$. Then we have that $\{x, \mathcal{T}_1^{i\beta}(x), \ldots, \mathcal{T}_l^{i\beta}(x)\} \subseteq U$ for every $0 \leq i \leq p-1$, so $x \in \bigcap_{0 \leq i \leq p-1}(U \cap (\mathcal{T}_1^{i\beta})^{-1}U \cap \ldots \cap (\mathcal{T}_l^{i\beta})^{-1}U) \neq \emptyset$, the conclusion.

Let that Theorem 5.2 holds for some $l \in \mathbb{N}$. We will show that Theorem 5.1 holds for $l+1$. Let $(X, \mathcal{T}_1, \ldots, \mathcal{T}_{l+1})$ a $\mathbb{Z}(p^\infty)$-dynamical system, where $\mathcal{T}_i$ is defined from the commuting homeomorphisms $\{T_{i,n}\}_{n\in\mathbb{N}}$ of X for $1 \leq i \leq l+1$ and $k \in \mathbb{N}$. Without lose of generality we can suppose that $(X, \mathcal{T}_1, \ldots, \mathcal{T}_{l+1})$ is minimal (or else we replace $(X, \mathcal{T}_1, \ldots, \mathcal{T}_{l+1})$ with a minimal subsystem). Let U_0 a non-empty open set with $diam(U_0) := \sup\{d(x,y) :\ x, y \in U_0\} < \frac{\varepsilon}{2}$. According to Theorem 5.2 (for the minimal system $(X, \mathcal{T}_1\mathcal{T}_{l+1}^{-1}, \ldots, \mathcal{T}_l\mathcal{T}_{l+1}^{-1}, \mathcal{T}_{l+1})$) there exists $\beta_1 \in \mathbb{Z}(p^\infty)^*$ with $k < \min dom(\beta_1)$ such that

$$B_0 := \bigcap_{0 \leq i \leq p-1} (U_0 \cap \bigcap_{s=1}^{l} [\mathcal{T}_s^{i\beta_1}(\mathcal{T}_{l+1}^{i\beta_1})^{-1}]^{-1} U_0) \neq \emptyset.$$

Let U_1 a non-empty open set with $diam(U_1) < \frac{\varepsilon}{2}$ such that
$U_1 \subseteq \bigcap_{0 \leq i \leq p-1} (\mathcal{T}_{l+1}^{i\beta_1})^{-1} B_0 = \bigcap_{0 \leq i \leq p-1} \bigcap_{s=1}^{l+1} (\mathcal{T}_s^{i\beta_1})^{-1} U_0$.
Suppose that for $m \in \mathbb{N}$ we have chosen $U_1, \ldots, U_m$ non-empty, open sets with $diam(U_q) < \frac{\varepsilon}{2}$ for every $1 \leq q \leq m$, such that

$$(**)\ \ U_j \subseteq \bigcap_{0 \leq i \leq p-1} \bigcap_{s=1}^{l+1} (\mathcal{T}_s^{i(\beta_j + \ldots + \beta_{q+1})})^{-1} U_q$$

for every $0 \leq q < j \leq m$, with $\beta_{j-1} \prec \beta_j$ for $2 \leq j \leq m$. From Theorem 5.2 there exist $\beta_{m+1} \in \mathbb{Z}(p^\infty)^*$ with $\beta_m \prec \beta_{m+1}$ such that

$$B_m := \bigcap_{0 \leq i \leq p-1} (U_m \cap \bigcap_{s=1}^{l} [\mathcal{T}_s^{i\beta_{m+1}}(\mathcal{T}_{l+1}^{i\beta_{m+1}})^{-1}]^{-1} U_m) \neq \emptyset.$$

Let U_{m+1} a non-empty open set with $diam(U_m) < \frac{\varepsilon}{2}$ and $U_{m+1} \subseteq \bigcap_{0\le i\le p-1} (\mathcal{T}_{l+1}^{i\beta_{m+1}})^{-1} B_m = \bigcap_{0\le i\le p-1} \bigcap_{s=1}^{l+1} (\mathcal{T}_s^{i\beta_{m+1}})^{-1} U_m$. Using this and $(**)$ for $j = m$ we have that for every $0 \le q \le m$,

$$U_{m+1} \subseteq \bigcap_{0\le i\le p-1} \bigcap_{s=1}^{l+1} (\mathcal{T}_s^{i(\beta_{m+1}+\ldots+\beta_{q+1})})^{-1} U_q.$$

Inductively we can suppose that we have sequences $(U_n)_{n\in\mathbb{N}\cup\{0\}}$ and $(\beta_n)_{n\in\mathbb{N}}$ with $\beta_n \prec \beta_{n+1}$ for every $n \in \mathbb{N}$, such that $(**)$ holds for every $m \in \mathbb{N}$, with $\beta_j + \ldots + \beta_{q+1} \in \mathbb{Z}(p^\infty)^*$ for every $0 \le q < j \le m,\ m \in \mathbb{N}$. For every $n \in \mathbb{N} \cup \{0\}$ let $x_n \in U_n$. Since X is sequential compact there exists $i_0 < j_0$ such that $d(x_{i_0}, x_{j_0}) < \frac{\varepsilon}{2}$. According to $(**)$, if we set $\beta = \beta_{j_0} + \ldots + \beta_{i_0+1} \in \mathbb{Z}(p^\infty)^*$ we have that $\{\mathcal{T}_1^{i\beta}(x_{j_0}), \ldots, \mathcal{T}_{l+1}^{i\beta}(x_{j_0})\} \subseteq U_{i_0}$ for every $0 \le i \le p-1$. Also, $x_{i_0} \in U_{i_0}$, $d(x_{i_0}, x_{j_0}) < \frac{\varepsilon}{2}$ and $diam(U_{i_0}) < \frac{\varepsilon}{2}$, thus for $x = x_{j_0}$, we have that

$$\mathcal{T}_j^{i\beta}(x) \in B(x, \varepsilon) \text{ for every } 0 \le i \le p-1,\ 1 \le j \le l+1.$$

The proof is complete. $\square$

We have already seen that Theorem 3.2 implies a partition theorem for the set $\mathbb{Z}(p^\infty)$ (Theorem 3.3). Using Theorem 5.1 we will state and prove, in Theorem 5.3, a partition theorem for the set $\mathbb{Z}(p^\infty)^l,\ l \in \mathbb{N}$, which is the multidimensional version of Theorem 3.3.

Theorem 5.3. *Let $l \in \mathbb{N}$ and k be an arbitrary real number. If $\mathbb{Z}(p^\infty)^l = Z_1 \cup \ldots \cup Z_r$, $r \in \mathbb{N}$, then there exists $1 \le i_0 \le r$ such that if $F \in [\mathbb{Z}(p^\infty)^l]^{<\omega}_{>0}$, then there exists $\alpha \in \mathbb{Z}(p^\infty)^l$ and $\beta \in \mathbb{Z}(p^\infty)^*$ with $k < \min dom(\beta)$ such that $\alpha + i\beta F \subseteq Z_{i_0}$ for every $0 \le i \le p-1$.*

Proof. Let $\mathbb{Z}(p^\infty)^l = Z_1 \cup \ldots \cup Z_r, r \in \mathbb{N}$. It suffices to produce the set Z_j for a given configuration F. For since there are only finite many possibilities for Z_j and since a sequence F_n may be chosen where each contains all the preceding ones and any F is contained in one of them, a set Z_j that occurs for infinitely many F_n will work for all F. That would be the desired Z_{i_0}. So we assume that a finite subset of $\mathbb{Z}(p^\infty)^l$, $F = \{\widetilde{e}_1, \ldots, \widetilde{e}_m\}$ is given. Let $\Omega = \{1, \ldots, r\}^{\mathbb{Z}(p^\infty)^l}$ and enumerate $\mathbb{Z}(p^\infty) = \bigcup_{n\in\mathbb{N}} \{q_n\}$ with $q_1 = 0$. Then Ω becomes a compact metric space if we define a metric by:

$$d(\omega, \omega') = \inf\{\tfrac{1}{t} :\ \omega(q_{i_1}, \ldots, q_{i_l}) = \omega'(q_{i_1}, \ldots, q_{i_l}) \text{ for } 1 \le i_1, \ldots, i_l < t\}.$$

For $1 \le j \le m$ and $\widetilde{q} \in \mathbb{Z}(p^\infty)^l$, let for $T_{j,n}\omega(\widetilde{q}) = \omega(\widetilde{q} + p^{-n}\widetilde{e}_i)$, $n \in \mathbb{N}$. For $1 \le i \le m$ we form the $\mathbb{Z}(p^\infty)$-indexed family $\mathcal{T}_i$ which is defined from the commuting homeomorphisms

$\{T_{j,n}\}_{n\in\mathbb{N}}$. We define a specific point $\omega \in \Omega$ by $\omega(\widetilde{q}) = i \;\Leftrightarrow\; \widetilde{q} \in Z_i$, and $\widetilde{q} \notin Z_j$ for $j < i$ and let

$$X = \overline{\{\mathcal{T}_1^{s_{1,1}} \dots \mathcal{T}_1^{s_{1,l_1}} \dots \mathcal{T}_m^{s_{m,1}} \dots \mathcal{T}_m^{s_{m,l_m}}\omega,\ s_{i,j} \in \mathbb{Z}(p^\infty),\ l_i \in \mathbb{N},\ 1 \le j \le l_i,\ 1 \le i \le m\}}.$$

Then, $(X, \mathcal{T}_1, \dots, \mathcal{T}_m)$ is a $\mathbb{Z}(p^\infty)$-dynamical system, so, according to Theorem 5.1 (for $\varepsilon = 1$) there exists $\beta \in \mathbb{Z}(p^\infty)^*$ with $k < \min dom(\beta)$ and $x_0 \in X$ such that $d(\mathcal{T}_j^{i\beta}(x_0), x_0) < 1$ for every $0 \le i \le p-1,\ \ 1 \le j \le m$. Thus

$$(*) \quad x_0((0, \dots, 0)) = x_0(i\beta\widetilde{e}_1) = \dots = x_0(i\beta\widetilde{e}_m) \text{ for every } 0 \le i \le p-1.$$

$x_0 \in X$, so it is arbitrary close to some translate of ω, $\mathcal{T}_1^{s_{1,1}} \dots \mathcal{T}_1^{s_{1,l_1}} \dots \mathcal{T}_m^{s_{m,1}} \dots \mathcal{T}_m^{s_{m,l_m}}\omega$, for some $s_{i,j} \in \mathbb{Z}(p^\infty),\ l_i \in \mathbb{N},\ 1 \le j \le l_i,\ 1 \le i \le m$.

Let $\widetilde{\alpha} = (s_{1,1} + \dots + s_{1,l_1})\widetilde{e}_1 + \dots + (s_{m,1} + \dots + s_{m,l_m})\widetilde{e}_m$. It follows from $(*)$ that

$$\omega(\widetilde{\alpha}) = \omega(\widetilde{\alpha} + i\beta\widetilde{e}_1) = \dots = \omega(\widetilde{\alpha} + i\beta\widetilde{e}_m) \text{ for every } 0 \le i \le p-1,$$

so, we have $\widetilde{\alpha} + i\beta F \subseteq Q_{\omega(\widetilde{\alpha})}$ for every $0 \le i \le p-1$. □

Remark 5.4. According to the proof of the previous theorem, we see that we cannot prove Theorem 5.1 or 5.2 from Theorem 5.3 (proving simultaneously the equivalence of these theorems), since we cannot control the domain of the coordinates of $\alpha \in \mathbb{Z}(p^\infty)^l$ for we have in general that $\mathcal{T}^{q_1+q_2} \neq \mathcal{T}^{q_1}\mathcal{T}^{q_2}$ (as we already noticed in Remark 2.4).

Definition 5.5. Let $l \in \mathbb{N}$ and k be an arbitrary real number. We say that the subset $B \subseteq \mathbb{Z}(p^\infty)^l$ is a *ZpVDW(l,k)*-set if for every $F \in [\mathbb{Z}(p^\infty)^l]^{<\omega}_{>0}$ there exist $\alpha \in \mathbb{Z}(p^\infty)^l$ and $\beta \in \mathbb{Z}(p^\infty)^*$ with $k < \min dom(\beta)$ such that $\alpha + i\beta F \subseteq B$ for every $0 \le i \le p-1$.

We will now prove that syndetic sets belongs to the previous family.

Proposition 5.6. *Let $l \in \mathbb{N}$ and k be an arbitrary real number. If E is a syndetic subset of $\mathbb{Z}(p^\infty)^l$, then E is a ZpVDW(l,k)-set.*

Proof. Let E be a syndetic subset of $\mathbb{Z}(p^\infty)^l$ and k be an arbitrary real number. Then, $\mathbb{Z}(p^\infty)^l = \bigcup_{x\in F}(E + x)$ for some $F \in [\mathbb{Z}(p^\infty)^l]^{<\omega}_{>0}$. According to Theorem 5.3 there exists $x_0 \in F$ such that $E + x_0$ is a ZpVDW(l,k)-set. So, E is a ZpVDW(l,k)-set, since this property is translation invariant. □

6. A Furstenberg-Weiss type Theorem for $\mathbb{Z}(p^\infty)$-dynamical systems

In this section we will prove (in Theorem 6.1) an analogous result of Theorem 0.1 for $\mathbb{Z}(p^\infty)$-dynamical systems, namely, we prove that if $(X, \mathcal{T}_1, \ldots, \mathcal{T}_l)$ is a $\mathbb{Z}(p^\infty)$-dynamical system and k is an arbitrary real number that there exist $x \in X$ and a sequence $(\beta_n)_{n\in\mathbb{N}} \subseteq \mathbb{Z}(p^\infty)^*$ with $k < \min dom(\beta_1),\ \beta_n \prec \beta_{n+1}$ for every $n \in \mathbb{N}$ such that $\mathcal{T}_j^{i\beta_n}(x) \to x$ for every $0 \le i \le p-1$ simultaneously for $1 \le j \le l$ (we call these points *multiple* $\mathbb{Z}(p^\infty)$-*recurrent points*). Moreover we prove that Theorem 6.1 is equivalent to Theorems 5.1 and 5.2 and also that the multiple $\mathbb{Z}(p^\infty)$-recurrent points consist a residual subset of X (Definition 6.2).

At this point (as in [Fu] for the analogous dynamical systems in $\mathbb{N}$ or $\mathbb{Z}$ and [K] for the dynamical systems in $\mathbb{Q}$) observe that if the condition of commutativity is omitted, the conclusion of this result need not holt. It may still happen that the return times of any point are disjoint for the various transformations. For example, let $X = \{-1, 1\}^{\mathbb{Z}(p^\infty)}$ and $\mathcal{T}$ be the $\mathbb{Z}(p^\infty)$-system which is defined from $\{T_n\}_{n\in\mathbb{N}}$ with

$$T_n\omega(q) = \omega(q + p^{-n}),\ n \in \mathbb{N}.$$

Let $R : X \to X$ with $R(\omega(q)) = \omega(q)$ if $q = 0$ and $R(\omega(q)) = -\omega(q)$ if $q \neq 0$ and let $S_n = RT_nR,\ n \in \mathbb{N}$. Then, if $\mathcal{S}$ is the $\mathbb{Z}(p^\infty)$-system which is defined from $\{S_n\}_{n\in\mathbb{N}}$, we have that $\mathcal{S}^q = R\mathcal{T}^qR$ for every $q \in \mathbb{Z}(p^\infty)$. $\mathcal{T}^q\omega$ close to ω implies that $\omega(q) = \omega(0)$. $\mathcal{S}^q\omega$ close to ω implies that $\mathcal{T}^qR\omega$ is close to $R\omega$, so $R\omega(q) = R\omega(0)$, thus $-\omega(q) = \omega(0)$ if $q \neq 0$. We have that $\mathcal{T}^q\omega$ and $\mathcal{S}^q\omega$ cannot be simultaneously close to ω.

Let $(X, \mathcal{T}_1, \ldots, \mathcal{T}_l),\ l \in \mathbb{N}$ be a $\mathbb{Z}(p^\infty)$-dynamical system. We want to find a point in the space X which returns close to itself for the same power through the action of $\mathcal{T}_1, \ldots, \mathcal{T}_l$.

The analogous of Theorem 0.1 related to the set $\mathbb{Z}(p^\infty)$ is the following.

Theorem 6.1. *Let $(X, \mathcal{T}_1, \ldots, \mathcal{T}_l)$ be a $\mathbb{Z}(p^\infty)$-dynamical system and k be an arbitrary real number. There exists a $x \in X$ and a sequence $(\beta_n)_{n\in\mathbb{N}} \subseteq \mathbb{Z}(p^\infty)^*$ with $k <$ $\min dom(\beta_1),\ \beta_n \prec \beta_{n+1}$ for every $n \in \mathbb{N}$ such that $\mathcal{T}_j^{i\beta_n}(x) \to x$ for every $0 \le i \le p-1$ simultaneously for $1 \le j \le l$.*

Proof. For every $s > 0$ let

$$\begin{aligned} F_s = \{x \in X : {} & \text{there exists } \beta \in \mathbb{Z}(p^\infty)^* \text{ with } k < \min dom(\beta) \text{ such that} \\ & d(\mathcal{T}_j^{i\beta}(x), x) < \tfrac{1}{s} \text{ for every } 0 \le i \le p-1,\ 1 \le j \le l\}. \end{aligned}$$

If the conclusion is not true, then $X = \bigcup_{n\in\mathbb{N}}(X \setminus F_n)$. We claim that for every $n \in \mathbb{N}$ we have that $(X \setminus F_n)^\circ = \emptyset$, a contradiction according to Baire's Category Theorem, since every $X \setminus F_n$ is closed.

Suppose that $(X\setminus F_{n_0})^\circ \neq \emptyset$ for some $n_0 \in \mathbb{N}$. Without loss of generality we can suppose that $(X, \mathcal{T}_1, \ldots, \mathcal{T}_l)$ is minimal. For $1 \leq i \leq l$, let $\mathcal{T}_i$ be defined from the commuting homeomorphisms $\{T_{i,n}\}_{n\in\mathbb{N}}$ of X. If G is the group of homeomorphisms generated by $\{T_{i,n}\}_{n\in\mathbb{N}}$, $1 \leq i \leq l$, then $X = S_1^{-1}(X \setminus F_{n_0})^\circ \cup \ldots \cup S_m^{-1}(X \setminus F_{n_0})^\circ$ for some $S_1, \ldots, S_m \in G, m \in \mathbb{N}$. Choose $\delta > 0$ such that if $y, z \in X$ with $d(y, z) < \delta$ then $d(S_i(y), S_i(z)) < \frac{1}{n_0}$ for every $1 \leq i \leq m$.

We claim that if $x \in S_q^{-1}(X \setminus F_{n_0})^\circ$ for some $1 \leq q \leq m$, then $x \in X \setminus F_{\frac{1}{\delta}}$. Indeed, if there is some $\beta \in \mathbb{Z}(p^\infty)^*$ with $k < \min dom(\beta)$ such that $d(\mathcal{T}_j^{i\beta}(x), x) < \delta$ for every $0 \leq i \leq p-1$, $1 \leq j \leq l$, then $d(S_q(\mathcal{T}_j^{i\beta}(x)), S_q(x)) = d(\mathcal{T}_j^{i\beta}(S_q(x)), S_q(x)) < \frac{1}{n_0}$ for every $0 \leq i \leq p-1$, $1 \leq j \leq l$, since S_q commutes with $\{T_{j,n}\}_{n\in\mathbb{N}}, 1 \leq j \leq l$, with $S_q(x) \in (X \setminus F_{n_0})^\circ$, a contradiction.

Since every $x \in X$ is in $S_q^{-1}(X \setminus F_{n_0})^\circ$ for some $1 \leq q \leq m$, we have proved that $x \in X\setminus F_{\frac{1}{\delta}}$ for every $x \in X$, so, $X\setminus F_{\frac{1}{\delta}} = X$ a contradiction according to Theorem 5.1. □

Definition 6.2. A subset $U \subseteq X$ is called *residual* if it contains an enumerable intersection of dense sets.

Theorem 6.1 gives the following.

Proposition 6.3. *If $(X, \mathcal{T}_1, \ldots, \mathcal{T}_l)$ is a minimal $\mathbb{Z}(p^\infty)$-dynamical system, then the set of multiple $\mathbb{Z}(p^\infty)$-recurrent points of X is residual.*

Proof. It follows from Theorem 6.1, since for every $n \in \mathbb{N}$ the set F_n is dense, open and $\emptyset \neq F_n \subseteq \{$multiple $\mathbb{Z}(p^\infty)$-recurrent points$\}$. □

Proposition 6.4. *Theorems 5.1, 5.2 and 6.1 are equivalent.*

Proof. We have already seen that Theorems 5.1 and 5.2 are equivalent and that Theorem 6.1 follows from Theorem 5.1. Let $l \in \mathbb{N}, (X, \mathcal{T}_1, \ldots, \mathcal{T}_l, R)$ be a minimal $\mathbb{Z}(p^\infty)$-dynamical system and k be an arbitrary real number. According to the proof of Theorem 6.1, if U is a non-empty set in X then there exists a $x \in U$ and a sequence $(\beta_n)_{n\in\mathbb{N}} \subseteq \mathbb{Z}(p^\infty)^*$ with $k < \min dom(\beta_1)$, $\beta_n \prec \beta_{n+1}$ for every $n \in \mathbb{N}$ such that $\mathcal{T}_j^{i\beta_n}(x) \to x$ for every $0 \leq i \leq p-1$, $1 \leq j \leq l$. So, there exists $n_0 \in \mathbb{N}$ such that $\mathcal{T}_j^{i\beta_{n_0}}(x) \in U$ for every $0 \leq i \leq p-1$, $1 \leq j \leq l$.

Hence $\bigcap_{0\leq i\leq p-1}(U \cap (\mathcal{T}_1^{i\beta_{n_0}})^{-1}U \cap \ldots \cap (\mathcal{T}_l^{i\beta_{n_0}})^{-1}U) \neq \emptyset$, the conclusion. □

Definition 6.5. Let k be an arbitrary real number. A subset $A \subseteq \mathbb{Z}(p^\infty)$ is called *ZpIP(k)-set* if there exists a sequence $(\beta_n)_{n\in\mathbb{N}} \subseteq \mathbb{Z}(p^\infty)^*$ with $k < \min dom(\beta_1)$, $\beta_n \prec \beta_{n+1}$ for every $n \in \mathbb{N}$ such that A consists of the numbers $i\beta_j,\ 0 \leq i \leq p-1$ together with all the finite sums in the form $i(\beta_{j_1} + \ldots + \beta_{j_s}),\ 0 \leq i \leq p-1$ with $j_1 < \ldots < j_s$.

Proposition 6.6. *Let $(X, \mathcal{T}_1, \ldots, \mathcal{T}_l)$ be a $\mathbb{Z}(p^\infty)$-dynamical system, k be an arbitrary real number and x_0 a multiple $\mathbb{Z}(p^\infty)$-recurrent point of X. Then, for every $\delta > 0$ the set*

$$R_\delta = \{q \in \mathbb{Z}(p^\infty) :\ d(\mathcal{T}_j^q(x_0), x_0) < \delta \textit{ for every } 1 \leq j \leq l\}$$

contains a ZpIP(k)-set.

Proof. Let $\delta > 0$ and x_0 a point which satisfies the conclusion of Theorem 6.1. According to Theorem 6.1 there exist $\beta_1 \in \mathbb{Z}(p^\infty)^*$ with $k < \min dom(\beta_1)$ such that

(1) $\quad d(\mathcal{T}_j^{i\beta_1}(x_0), x_0) < \delta$ for every $0 \leq i \leq p-1,\ 1 \leq j \leq l$.

Let $0 < \delta_2 \leq \delta$ such that

(2) $\quad d(x, x_0) < \delta_2 \Rightarrow d(\mathcal{T}_j^{i\beta_1}(x), x_0) < \delta$ for every $0 \leq i \leq p-1,\ 1 \leq j \leq l$.

According to Theorem 6.1 there exist $\beta_2 \in \mathbb{Z}(p^\infty)^*$ with $\beta_1 \prec \beta_2$ such that

(3) $\quad d(\mathcal{T}_j^{i\beta_2}(x_0), x_0) < \delta_2$ for every $0 \leq i \leq p-1,\ 1 \leq j \leq l$.

The conditions $(1), (2)$ and (3) ensures that

$(*)$ $\quad d(\mathcal{T}_j^{m}(x_0), x_0) < \delta$, where $m = i\beta_1$ or $i\beta_2$ or $i(\beta_1 + \beta_2)$ for every $0 \leq i \leq p-1$, $1 \leq j \leq l$.

Assume that we have found $\beta_1, \ldots, \beta_n$ with $\beta_s \prec \beta_{s+1}$ for every $s = 1, \ldots, n-1$ such that $(*)$ holds for $m = i(\beta_{j_1} + \ldots + \beta_{j_s})$ for every $0 \leq i \leq p-1,\ 1 \leq j \leq l$ with $j_1 < \ldots < j_s \leq n$. Let $\delta_{n+1} \leq \delta$ such that $d(x, x_0) < \delta_{n+1} \ \Rightarrow\ d(\mathcal{T}_j^m(x), x_0) < \delta$ for the previous $m,\ 1 \leq j \leq l$. According to Theorem 6.1 there exist $\beta_{n+1} \in \mathbb{Z}(p^\infty)^*$ with $\beta_n \prec \beta_{n+1}$ such that $d(\mathcal{T}_j^{i\beta_{n+1}}(x_0), x_0) < \delta_{n+1}$ for every $0 \leq i \leq p-1,\ 1 \leq j \leq l$. Then, $(*)$ holds if we replace m with $m + i\beta_{n+1}$ or $i\beta_{n+1}$ for every $0 \leq i \leq p-1$. Inductively, we have that the set $R = \{i(\beta_{j_1} + \ldots + \beta_{j_s}),\ 0 \leq i \leq p-1 \text{ with } j_1 < \ldots < j_s\}$ is contained in R_δ. $\square$

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